Continue today with the Lie derivative, which is like the convective derivative of ordinary tensor analysis, but generalized to arbitrary manifolds.

Given a manifold $M$, a vector field $X \in \mathfrak{X}(M)$, with advance map $\Phi_t : M \to M$. Illustrate Lie derivative first with scalar fields, where $L_X : \mathcal{F}(M) \to \mathcal{F}(M)$ (i.e. $L_X$ is the Lie derivative along vector field $X$.)

Let $x_0 \in M$ and $x_t = \Phi_t(x_0)$. We think of $t$ as small (we will be interested in the limit $t \to 0$). For $f \in \mathcal{F}(M)$, define

$$(L_X f)(x_0) = \lim_{t \to 0} \frac{1}{t} \left[ f(x_t) - f(x_0) \right]$$

at integral curve of $X$.

It's pretty obvious from this formula that $L_X f = Xf$, since the vector $X|_{x_0}$ is the small displacement $x_0 \to x_t$ in small time $t$. So:

Thus, the Lie derivative of a scalar is the obvious generalization of the convective derivative to an arbitrary manifold,

$$L_X f = Xf = \sum_i X_i \frac{\partial f}{\partial x^i}.$$ (Think $\nabla \cdot v f$).

But transform equiv. above:

$$(L_X f)(x_0) = \lim_{t \to 0} \frac{1}{t} \left[ f(\Phi_t(x_0)) - f(x_0) \right]$$

$$= \lim_{t \to 0} \frac{1}{t} \left[ (\Phi_t^X f)(x_0) - f(x_0) \right]$$
\[
\begin{align*}
\lim_{t \to 0} \frac{1}{t} \left( (e^{t\Phi_t^x} - 1) f(x) \right) \\
= \left( \frac{d}{dt} \bigg|_{t=0} \Phi_t^x \right) f(x).
\end{align*}
\]

But recall,
\[
\Phi_t^x = e^{tx}
\]
when acting on scalars, so we find again, \( \frac{d\Phi_t^x}{dt} \bigg|_{t=0} = x \), \( L_x f = xf \).

Now generalize to other differential geometric objects, like vector fields. Now we want to define \( L_x \) as an operator: \( \mathcal{X}(M) \to \mathcal{X}(M) \).

Let \( Y \in \mathcal{X}(M) \) (now we have 2 vector fields, \( X \) and \( Y \)), and we wish to define \( L_X Y \). The idea is the same as above, we wish to compare \( Y \) at \( x_1 \) with \( Y \) at \( x_0 \) to see how much \( Y \) has changed along the integral curves of \( X \). But we cannot just subtract \( Y(x_1) - Y(x_0) \), these vectors belong to two different tangent spaces \( T_{x_0} M \) and \( T_{x_1} M \) without any natural identification. However, we can "pull-back" \( Y(x_1) \) to point \( x_0 \) using the flow (mapping both base and tip of arrow by \( \Phi_t^{-1} \)). Note that the pull-back of a vector field is defined in this case because \( \Phi_t \) is invertible; the pull-back is the inverse of the tangent map \( \Phi_t^* \).

So, we define...
\[ \mathfrak{L}_x Y = \left( \frac{d}{dt} \bigg|_{t=0} \Phi_{t*}^{-1} \right) Y, \text{ or} \]

\[
(\mathfrak{L}_x Y)(x_0) = \lim_{t \to 0^+} \left[ (\Phi_{t*}^{-1}) Y(x_0) - Y(x_0) \right]
\]

In components,

\[
(\Phi_{t*}^{-1} Y)^i(x_0) = \frac{\partial x^i_0}{\partial x'^j} Y^j(x_1).
\]

To get \( x_1 \) as a fun of \( x_0, t \), we solve the ODE's,

\[
\frac{dx^i}{dt} = X^i(x)
\]

in power series in \( t \),

\[
x^i_1 = x^i_0 + t X^i(x_0) + \ldots
\]

or its inverse,

\[
x^i_0 = x^i_1 - t X^i(x_0) + \ldots
\]

so,

\[
\frac{\partial x^i_0}{\partial x'^j} = \delta^i_j - t x^i_{1,j} + \ldots
\]

and,

\[
Y^i(x_0) + t \left( X^j Y^i_{,j} - Y^j X^i_{,j} \right)
\]

so,

\[
(\mathfrak{L}_x Y)^i = X^j Y^i_{,j} - Y^j X^i_{,j}
\]

This is the Lie derivative of a vector field.

Similarly, you can define the Lie deriv. of a covector field.
\[ L_x \alpha = (\frac{d}{dt} \Phi_t^* \bigg|_{t=0}) \alpha. \]

If you work it out, you find (in components),
\[ (L_x \alpha)_i = x^i \alpha_{i,j} - \alpha_j x^j_i. \]

To define \( L_x \) on arbitrary tensors, we develop some general rules. First, \( L_x \) acts on a tensor product of tensors by the Leibnitz rule. An example will illustrate. Consider the tensor product of a covector with a vector (this is a type \((1,1)\) tensor): define
\[ L_x (\alpha \otimes Y) = \left( \frac{d}{dt} \bigg|_{t=0} [ (\Phi_t^* \alpha) \otimes (\Phi_t^* x^i Y_i) ] \right), \]

which is the obvious definition. But this is...

\[ (L_x \alpha) \otimes Y + \alpha \otimes (L_x Y), \]

(Leibnitz rule).

The same thing works on contractions. For example, the tensor \( \alpha \otimes Y \) has components,
\[ (\alpha \otimes Y)_i^j = \alpha_i Y^j. \]

If we contract (set \( i = j \) and sum), we get
\[ \alpha_i Y^i = \alpha(Y), \]  
a scalar.

Then we have
\[ L_x [ \alpha(Y) ] = (L_x \alpha)(Y) + \alpha (L_x Y) \]
\[ = X(\alpha(Y)). \]  
Can use this to calculate \( L_x \alpha \) in components, supposing that we know what \( L_x \) does to scalars.
and vector fields.

Notice that a scalar multiplied by a tensor is a special case of a tensor product:

\[ f \otimes T = fT \quad \text{any } T, \ f \in F(M). \]

Therefore

\[ L_x (fT) = (L_x f) T + f (L_x T) = (xf) T + f (L_x T). \]

Since an arbitrary tensor can be written as linear combinations of scalars times tensor products of vector fields and covector fields, the Leibnitz rule suffices to compute the Lie derivative of any tensor.

Some more rules about \( L_x \). If \( f \in F(M) \), then

\[ L_x[f] = f L_x. \]

This is obvious since \( fX \) has some integral curves as \( X \), except the \( t \)-parametrization is scaled by \( f \). Hence \( \frac{d}{dt} \bigg|_{t=0} \) is scaled by \( f \).

The Lie derivative \( L_x Y \) is a special case with a special interpretation. Consider the flows associated with \( X, Y \), call them \( \Phi_t, \Psi_t \). These in general do not commute.

\[ \Phi_t \Psi_t \neq \Psi_t \Phi_t \]

\( \Phi_t \Psi_t \) don't agree. I small vector here, when \( \delta t \) small.
When $s,t$ are small, the difference in the endpoints must be a vector. However, since we cannot subtract points, to measure the difference between $\Psi_t \Phi_s x_0$ and $\Phi_s \Psi_t x_0$ we evaluate some scalar $f : M \rightarrow \mathbb{R}$ at the 2 points and subtract:

$$f(\Psi_t \Phi_s x_0) - f(\Phi_s \Psi_t x_0)$$

$$= \left( (\Psi_t \Phi_s)^* f \right)(x_0) - \left( (\Phi_s \Psi_t)^* f \right)(x_0)$$

$$= (\Phi_s^* \Psi_t^* f)(x_0) - (\Psi_t^* \Phi_s^* f)(x_0)$$

$$= \left( (\Phi_s^* \Psi_t^* - \Phi_s^* \Phi_s \Psi_t^* ) f \right)(x_0)$$

$$= e^s X e^t Y e^s X - e^t Y e^s X = (1 + sX + \frac{s^2 X^2}{2} + \ldots)(1 + tY + \frac{t^2 Y^2}{2} + \ldots) - (x \otimes y)$$

$$= 1 + (sX + tY) + \left( \frac{s^2}{2} X^2 + stXY + \frac{t^2}{2} Y^2 \right) + \ldots - (x \otimes y)$$

$$= st \left( XY - YX \right) + \ldots$$

Thus the leading term is the commutator of $X$ and $Y$ (regarded as maps: $f(M) \rightarrow \mathbb{R}$). Thus we have

$$\frac{\partial^2}{\partial s \partial t} \bigg|_{t=0} f(\Psi_t \Phi_s x_0) - f(\Phi_s \Psi_t x_0) = \left( [X,Y] f \right)(x_0),$$

for all $f \in \mathcal{F}(M)$. 
Now $XY$ is not a vector field (because it is a 2nd order operator), but it turns out that $XY - YX$ is a vector field (all 2nd deriv cancel), and, in fact,

$$L_X Y = [X,Y]$$

The (important) commutator has the following properties:

a) $[X,Y] = -[Y,X]$  

b) $[X,Y]$ linear in $X, Y$ (over $\mathbb{R}$)  

c) $[X,[Y,Z]] + [Z,X] + [X,Y] = 0$ (Jacobi).

The set $\mathfrak{X}(M)$ forms a Lie algebra. (of diffeomorphism group).

Some other properties of the commutator:

(a) $f_* [X,Y] = [f_* X, f_* Y]$ when $f: M \to N$ is a diffeomorphism  

(b) $L_{[X,Y]} = [L_X, L_Y]$  

$\Rightarrow$ lie algebra maps commute w. diffeomorphisms.

(c) is almost obvious; a diffeomorphism is an isomorphism of differentiable structure, integral curves mapped into integral curves etc.

Now we turn to differential forms. A diff. form of rank $r$ is a completely antisymmetric type $(0,r)$ tensor. Why antisymmetric? Because you need these for integrating over oriented, $r$-dimensional surfaces. Consider an example from 3D vector calculus. Let a small area element be specified by two small vectors $\overrightarrow{\xi}$ and $\overrightarrow{\eta}$. These might define a small element of a 2D surface.
Then let $\mathbf{F}$ be a flux vector (of mass, charge, etc., or maybe $\mathbf{F}=\mathbf{B}$ = magnetic field). Then the flux through parallelogram is

$$\mathbf{F} \cdot (\mathbf{\hat{e}} \times \mathbf{\hat{\eta}}).$$

Why $\mathbf{\hat{e}} \times \mathbf{\hat{\eta}}$ and not $\mathbf{\hat{\eta}} \times \mathbf{\hat{e}}$? Because we have to decide which side of the parallelogram is the "outward" oriented side (it's a convention, but the sign of the answer depends on it). So the area element is specified by $\mathbf{\hat{e}} \times \mathbf{\hat{\eta}}$, which is antisymmetric in the two vectors. And the value of the flux is the value of a linear operator that acts on area elements. It's like a covector (acts on vectors), except that it acts on 2 vectors (effectively area elements). Note, we can write

$$\mathbf{F} \cdot (\mathbf{\hat{e}} \times \mathbf{\hat{\eta}}) = \frac{1}{2} J_{ij} (\mathbf{\hat{e}}^i \mathbf{\hat{\eta}}^j - \mathbf{\hat{e}}^j \mathbf{\hat{\eta}}^i),$$

where $J_{ij} = \varepsilon_{ijk} J^k$. $J_{ij} = -J_{ji}$ are the components of a 2-form.

Special cases of $r$-forms:

$r=0$ is a scalar, or 0-form, considered to be antisymmetric in its nonexistent operands.

$r=1$ is a covector, or 1-form, considered to be antisymmetric in its one operand, $\alpha : \mathfrak{X}(M) \to \mathbb{R} \mathfrak{X}(M)$ (as a field).

$r=2$ is a 2-form, an antisymmetric tensor acting on two vector fields,

$$\omega : \mathfrak{X}(M) \otimes \mathfrak{X}(M) \to \mathfrak{X}(M),$$

$$\omega(X, Y) = -\omega(Y, X).$$
Cases \( r = 0, 1, 2 \)

In components:
- Let \( x \in \mathcal{M} \), \( x^i \) = coordinates of \( x \).
- Let \( e_i = \frac{\partial}{\partial x^i} \) = basis vectors of coordinate system.

A scalar \( f : \mathcal{M} \to \mathbb{R} \) has only one component, the value \( f(x) \) of \( f \) itself.

A covector or 1-form \( \alpha \) has components,

\[ \alpha_i(x) = \alpha(e_i)|_x. \]

A 2-form \( \omega \) has components,

\[ \omega_{ij}(x) = \omega(e_i, e_j)|_x = -\omega_{ji}(x). \]

The number of independent components of an \( r \)-form on an \( n \)-dimensional space is

\[
\binom{n}{r} = \begin{cases} 1 & r = 0 \\ n & r = 1 \\ \frac{n(n-1)}{2} & r = 2 \\ \vdots & \vdots \\ 1 & r = n. \end{cases}
\]

Another special case is an \( n \)-form, call it \( \Phi \). This is a completely antisymmetric map of \( n \) vectors to scalars,

\[ \Phi : \mathcal{T}(\mathcal{M}) \times \ldots \times \mathcal{T}(\mathcal{M}) \to \mathcal{F}(\mathcal{M}) \]

with components

\[ \Phi_{i_1 \ldots i_n}(x) = \text{completely antisymmetric in indices} \ (i_1, \ldots, i_n), \]

\[ = \Phi(e_{i_1}, \ldots, e_{i_n}). \]
Thus, a $n$-form on an $n$-dimensional manifold has components in any given chart that have the form

$$
\phi_{i_1...i_n}(x) = \sigma(x) \varepsilon_{i_1...i_n},
$$

where $\varepsilon_{i_1...i_n}$ is the Levi-Civita symbol, and $\sigma(x)$ is a scalar density. $\sigma(x)$ defines the only independent component of $\phi$.

We write the set of all smooth $r$-forms on $M$ as $\Omega^r(M)$. Thus, $\Omega^0(M) = \mathcal{F}(M)$, $\Omega^1(M) = \mathcal{X}(M)$, etc.

We consider $r$-forms with $r > n$ to be zero.

How to construct $r$-forms. One way is to take the exterior product of $r$ 1-forms. The exterior product is an antisymmetrized tensor product.

The exterior product of $r$ 1-forms is defined as follows.

Let $\alpha^1, ..., \alpha^r$ be 1-forms ($\alpha^i \in \Omega^1(M)$).

Then $\alpha^1 \wedge ... \wedge \alpha^r$ is an $r$-form, defined by its action on $r$ vector fields $X_1, ..., X_r$ by

$$
(\alpha^1 \wedge ... \wedge \alpha^r)(X_1, ..., X_r) = \sum_{P \in S_r} (-1)^P \alpha^1(X_{P_1}) \alpha^2(X_{P_2}) ... \alpha^r(X_{P_r})
$$

where $S_r$ is set of all permutations $P$ of $r$ objects. More precisely, $P$ is a bijection of the set $\{1, 2, ..., r\}$ to itself, $P_i = \text{value of } P \text{ acting on } i$ ($1 \leq i \leq r$). $(-1)^P$ is the parity of the permutation ($+1$ if even, $-1$ if odd).
can also write this as

\[(\alpha' \wedge \ldots \wedge \alpha^r)(x_1, \ldots, x_r) = \begin{vmatrix}
\alpha'(x_1) & \ldots & \alpha'(x_r) \\
\vdots & & \vdots \\
\alpha^r(x_1) & \ldots & \alpha^r(x_r)
\end{vmatrix}\]

Example: Let \(\alpha, \beta \in \Omega'(M)\), \(x, y \in \mathcal{X}(M)\)

\[(\alpha \wedge \beta)(x, y) = \begin{vmatrix}
\alpha(x) & \alpha(y) \\
\beta(x) & \beta(y)
\end{vmatrix} = \alpha(x)\beta(y) - \alpha(y)\beta(x).
\]

Properties:

1) \(\alpha' \wedge \ldots \wedge \alpha^r\) is completely antisymmetric,

\[\alpha^p \wedge \ldots \wedge \alpha^r = (-1)^p \alpha^1 \wedge \ldots \wedge \alpha^r.
\]

In particular, \(\alpha \wedge \beta = -\beta \wedge \alpha\) \(\quad (\alpha, \beta \in \Omega'(M)).\)

2) If \(\alpha^i = \alpha^j\) for any \(i \neq j\), then \(\alpha^1 \wedge \ldots \wedge \alpha^r = 0\).

A general \(r\)-form is not the exterior product of a set of \(r\) 1-forms, but can always be represented as a linear combination of such products. Example: Let \(A\) be an antisymmetric, \((0,2)\) tensor,

\[A = A_{\mu\nu} \, dx^\mu \otimes dx^\nu \quad (A_{\mu\nu} = -A_{\nu\mu})\]

\[= \frac{1}{2} \, A_{\mu\nu} \left[ dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu \right]\]

\[= \frac{1}{2} \, A_{\mu\nu} \, dx^\mu \wedge dx^\nu.\]
Thus, \( dx^\mu \wedge dx^\nu \ (\mu, \nu = 1, \ldots, n) \) is a basis of 2-forms on \( M \).

Similarly, for a general r-form,

\[
\omega = \frac{1}{r!} \omega_{\mu_1 \ldots \mu_r}(x) \, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}
\]

basis of r-forms.

Here we use summation convention, \( \sum \) implied only over indices in ascending order,

\[
\sum_{\mu_1 < \mu_2 < \ldots < \mu_r}
\]

You can drop the factor of \( \frac{1}{r!} \).

Now generalize the exterior product to arbitrary forms.

Let \( \alpha \in \Omega^r(M) \), \( \beta \in \Omega^s(M) \). Then \( \alpha \wedge \beta \in \Omega^{r+s}(M) \), defined by

\[
(\alpha \wedge \beta)(x_1, \ldots, x_{r+s}) = \frac{1}{r! s!} \sum_{\pi \in S_{r+s}} (-1)^{\pi} \alpha(x_{\pi_1}, \ldots, x_{\pi_r}) \beta(x_{\pi_{r+1}}, \ldots, x_{\pi_{r+s}}).
\]

Properties:

1) \( (\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) \) (Associative)

2) \( \alpha \wedge \beta = (-1)^{rs} \beta \wedge \alpha \).

Reason: \( (-1)^{rs} \) because need \( rs \) exchanges to swap order of factors.

Note: 2) implies \( \alpha \wedge \alpha = 0 \) when \( r = \text{odd} \).

Note special case, \( r = 0 \), \( \alpha = 0 \)-form \( \equiv f \). Then \( f \wedge \beta = f \beta \) (ord. mult.)

3) \( f^*(\alpha \wedge \beta) = f^* \alpha \wedge f^* \beta \) where \( f: M \to N \).