This $\mathbb{Z}_2$ is responsible for the line defect (vortex) in a nematic liquid, that can annihilate when it meets another such line defect, leaving behind no defect at all. [It's not that vortices of "opposite charge" are annihilating, rather, there is only one charge and it obeys the rule $1+1=0$.]

While on the subject of $\mathbb{RP}^n$, note special case $n=1$. Recall, $\mathbb{RP}^n$ is the sphere $S^n$ with antipodal points identified; equivalently, it is the $n$-dimensional disk $D^n$ (= the "northern hemisphere" of $S^n$) with antipodal points on the boundary identified. (The disk $D^n$ is region $r \leq 1$ in $n$-dim. space $\mathbb{R}^n$; it is the sphere $S^{n-1}$ plus all interior points.) So, for $n=1$, we get a circle with opposite points identified, or $D^1$, the 1-disk, which is a line segment with opposite endpoints identified:

$$\mathbb{RP}^1 = \frac{S^1}{\sim} = \frac{D^1}{\sim} = \frac{[0,1]}{\sim} = S^1$$

You might say the final circle is $\frac{1}{2}$ as big as the first one.

Now is a good time to comment on the relationship between classical rotations and spin rotations in QM. A classical rotation is an element of $\text{SO}(3)$, a linear map of $\mathbb{R}^3$ onto itself that preserves lengths, angles, and cross products. It takes 3 parameters to specify, some $R \in \text{SO}(3)$ (i.e., $\text{SO}(3)$ is a 3-dimensional manifold). These parameters can be specified in various ways. One is the Euler angles (often an ugly choice). Another is the axis-angle parameterization:

(continued on p. 11.)
Motivation for studying relationship between $SO(3)$ and $SU(2)$. Consider neutron or even evolution of spin $1/2$ particle in magnetic field $\mathbf{B} = \mathbf{B}(t)$ which we allow to be time-dependent. Define

$$\vec{\omega}(t) = g \frac{e}{2mc} \vec{B}(t)$$

a vector with dimensions of frequency ($g$ = g-factor of particle).

Let $\chi = (\psi_+\, \psi_-) \in \mathbb{C}^2$ be the usual spinor. The Schrödinger eqn is

$$i\hbar \frac{\partial \chi}{\partial t} = \vec{\omega}(t) \cdot \left( \frac{\hbar}{2} \vec{S} \right) \chi \quad (Qu)$$

where $\frac{\hbar}{2} \vec{S}$ is the spin operator. Let $\vec{S}(t)$ be the expectation value of the spin operator,

$$\vec{S}(t) = \langle \chi(t) | \frac{\hbar}{2} \vec{S} | \chi(t) \rangle \quad (Mc)$$

so that $\vec{S}$ is a c-number vector (not a vector of operators, $\vec{S} \in \mathbb{R}^3$).

Then

$$\frac{d\vec{S}}{dt} = \vec{\omega}(t) \times \vec{S} \quad (Cl)$$

$(Qu)$ is the "quantum eqn" and $(Cl)$ is the "classical" eqn. (classical in the sense that equations just like this occur in classical mechanics, they are the Euler equations). The solutions of $(Qu)$ and $(Cl)$ are

$$\chi(t) = U(t) \chi_0, \quad U(t) \in SU(2)$$

$$\vec{S}(t) = R(t) \vec{S}_0, \quad R(t) \in SO(3).$$

where $U(0) = 1$ (the $2 \times 2$ identity) and $R(0) = 1$ (the $3 \times 3$ identity).

The functions $U(t)$ and $R(t)$ are actually paths on the group manifolds $SU(2)$ and $SO(3)$. Let

$$\alpha: [0, T] \rightarrow SO(3)$$

$$\tilde{\alpha}: [0, T] \rightarrow SU(2)$$

$(T = \text{final time})$
be two paths in $SO(3)$ and $SU(2)$, where $\alpha(t)$ means $R(t)$ and $\bar{\alpha}(t)$ means $U(t)$, satisfying $\bar{\alpha}(0) = 1$, $\alpha(0) = I$. Picture on the group manifolds,

Consider the stmt: "If you rotate a neutron by 360°, it doesn't return to its original self but rather undergoes a phase change of -1. You have to rotate it by 720° to make it return to itself." Actually it is not the final value of the classical rotation $R(t)$ (or $\alpha(t)$) that determines the outcome, but rather the history. Here is a correct stmt:

Let $R(t) = \alpha(t) = I$ (at $t = T$). Then $\alpha: [0,T] \rightarrow SO(3)$ is a loop based at $I$. But $SO(3) = RP^3$ (topologically speaking), so there are two homotopy classes the loop $\alpha$ can be in, the trivial class or the nontrivial class, since $\pi_1(RP^3) = \mathbb{Z}_2$. Then

$$U(T) = \bar{\alpha}(T) = \begin{cases} +1 & \text{if } \alpha \in \text{trivial (contractible) class} \\ -1 & \text{if } \alpha \in \text{other class} \end{cases}$$

The final state of the neutron depends on the homotopy class of the loop $\alpha$ in $SO(3)$. In fact one may say that the existence of spin is related to this nontrivial homotopy group $\pi_1(SO(3)) = \mathbb{Z}_2$. 
There is an important map \( p : \text{SU}(2) \rightarrow \text{SO}(3) \) that occurs in this theory. (\( p \) stands for "projection.") It is defined by...

\[
R_{ij} = \frac{1}{2} \text{tr} \left( U^+ \sigma_i U \sigma_j \right), \quad \text{where } U \in \text{SU}(2),
\]

i.e., it defines a function \( R(U) \) or \( R = p(U) \). One can show that

\[
R(t) = p(U(t))
\]
in the spin problem, i.e., \( p \) maps the path \( \alpha(t) \) in \( \text{SU}(2) \) into \( \alpha(t) \) in \( \text{SO}(3) \). Note that \( p(U) = p(-U) \), so the inverse \( p^{-1}(R) \) of \( R \in \text{SO}(3) \) consists of 2 points \( U \) and \(-U\) (it turns out there are only these two).

\( p \) is a two-to-one projection.

\( \text{SU}(2) \) is an example of a double cover of \( \text{SO}(3) \). This is an example of a space \( M \) (\( \text{SO}(3) \)) and its covering space \( \tilde{M} \) (\( \text{SU}(2) \)). The projection \( p \) in the general case is a map

\( p : \tilde{M} \rightarrow M \) from the covering to the covered spaces. The path \( \tilde{\alpha}(t) \) defined above in \( \tilde{M} = \text{SU}(2) \) is called the lift of the path \( \alpha(t) = R(t) \) in \( M = \text{SO}(3) \). We mention all this (as yet) undefined terminology to give an example a preview of what will come.

Covering spaces don't have to be groups, but in this example they are, and there is extra structure because of that. For example, \( p : \text{SU}(2) \rightarrow \text{SO}(3) \) is a group homomorphism, with kernel \( \{1,-1\} \) (the image is all of \( \text{SO}(3) \)).
Take it as geometrically obvious that an arbitrary (proper) rotation can be written in axis-angle form:

$$\mathbf{R}(\hat{n}, \theta) = \mathbf{I} + \hat{n} \times + \hat{n} \times \hat{n} \times \mathbf{I} \quad \text{using right-hand rule.}$$

The parameterization is unique except when $\theta = 0$, where $\mathbf{R}(\hat{n}, 0) = \mathbf{I}$ for any $\hat{n}$, and at $\theta = \pi$, where

$$\mathbf{R}(\hat{n}, \pi) = \mathbf{R}(-\hat{n}, \pi)$$

(same rot'n)

So if we write $\hat{\theta} = \hat{n} \theta$, so that $\hat{\theta} \in \mathbb{R}^3$, then $\text{SO}(3)$ is identified with a sphere (the 3D, solid interior of a sphere in $\mathbb{R}^3$) out to a radius of $\pi$, including the surface ($S^2$) at $\theta = \pi$, but with antipodal points $(\hat{n}, \hat{n})$ on the surface identified. In other words,

$$\text{SO}(3) = \mathbb{RP}^3.$$

This is an example of a group manifold.

As for $\text{SU}(2)$, it is the set of $2 \times 2$, complex, unitary matrices with $\det = +1$:

$$U \in \text{SU}(2) \implies UU^\dagger = U^\dagger U = I$$

and $\det U = +1$.

The condition $UU^\dagger = U^\dagger U = I$ means that the rows and columns form pairs of orthonormal, complex, unit vectors (in $\mathbb{C}^2$). Write

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad a, b, c, d \in \mathbb{C},$$
Because of the conditions \( uu^* = I \), \( \det u = \pm 1 \), the 4 complex components of \( u \in SU(2) \) satisfy certain constraints, and \( u \) can be written in terms of 4 real parameters \((x_0, x_1, x_2, x_3) = (x_0, \vec{x})\),

\[
U = x_0 I - i \vec{x} \cdot \tilde{\sigma} = 
\begin{pmatrix}
    x_0 - i x_3 & -x_2 - i x_1 \\
    x_2 - i x_1 & x_0 + i x_3
\end{pmatrix}
\]

where \( x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \). The \((x_0, x_1, x_2, x_3)\) are the Cayley-Klein parameters, and they show that topologically

\[ SU(2) = S^3. \]

The relation between \( SU(2) = S^3 \) and \( SO(3) \cong \mathbb{R}P^3 \) is just the identification of antipodal points \( U \) and \(-U\) in \( S^3 \) with a single element of \( SO(3) \cong \mathbb{R}P^3 \):

\[ \text{surface supposed to represent } S^3 \text{ in } \mathbb{R}^4 \]

So \( SO(3) \) can be thought of as the "northern-hemisphere" with antipodal points on the "equator" \((S^2)\) identified. This is the solid ball picture of \( SO(3) \) (in \( \Theta \) coordinates).

\[ SU(2) \text{ is said to be a "double cover" of } SO(3). \text{ This is an example of a covering space. Here is another, simpler, example.} \]

Let \( p: \mathbb{R} \rightarrow S^1 \) (\( p = "projection"\)) be the map defined by \( p(x) = e^{ix} \), where \( S^1 \) is identified with the unit circle in the complex plane. \( \mathbb{R} \) can be seen as "wrapping around" \( S^1. \)