Notes on deriving Hartle's Eqs. (9.45) and (9.46).

The algebra is messy if you don't do it right. Start with

\[ (1 - \frac{2M}{r}) \frac{dt}{d\tau} = e \quad (9.21) \]

\[ r^2 \frac{d\phi}{d\tau} = l \quad (9.22) \text{ with } \theta = \pi/2 \]

\[ e^2 = \left(1 - \frac{2M}{r}\right) \left(1 + \frac{\dot{r}^2}{r^2}\right) + \left(\frac{d\tau}{d\tau}\right)^2 \quad (A) \]

This last eqn is an intermediate step in deriving (9.25). It goes like this. Use

\[ -1 = \sum \cdot \sum = - \left(1 - \frac{2M}{r}\right) \left(\frac{dt}{d\tau}\right)^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} \left(\frac{d\tau}{d\tau}\right)^2 + r^2 \left(\frac{d\phi}{d\tau}\right)^2. \]

Substitute (9.21) and (9.22) (see above), and get \( (\text{get rid of } \frac{dt}{d\tau}, \frac{d\phi}{d\tau}) \):

\[ -1 = - \frac{e^2}{\left(1 - \frac{2M}{r}\right)} + \frac{1}{\left(1 - \frac{2M}{r}\right)} \left(\frac{d\tau}{d\tau}\right)^2 + \frac{\dot{r}^2}{l^2} \quad (B) \]

or

\[ \frac{e^2}{1 - \frac{2M}{r}} = \frac{1}{1 - \frac{2M}{r}} \left(\frac{dt}{d\tau}\right)^2 + \left(1 + \frac{\dot{r}^2}{l^2}\right). \quad (C) \]

Multiply by \( 1 - \frac{2M}{r} \) and get (A) above. Rewriting (A) slightly gives

\[ e = \frac{e^2 - 1}{2} = \frac{1}{2} \left(\frac{dt}{d\tau}\right)^2 + V_{\text{eff}}(r), \quad (D) \]
where \( V_{\text{eff}} (r) = - \frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{\ell^2 M}{r^3} \).  \( \text{(E)} \)

This is (9.26) and (9.28).

Now in a circular orbit, \( \left( \frac{dr}{dt} \right) = 0 \) and \( V_{\text{eff}} (r) = 0 \).

So,
\[
e^2 = \left( 1 - \frac{2M}{r} \right) \left( 1 + \frac{\ell^2}{r^2} \right) \]
\[
\text{in circular orbit.} \quad \text{(F)}
\]

and \[
\frac{M}{r^2} - \frac{\ell^2}{r^3} + \frac{3\ell^2 M}{r^4} = 0 \]
\[
\text{in circular orbit.} \quad \text{(G)}
\]

To get the book's (9.45), solve (G) for \( \ell^2 \). Muot by \( r^2 \),
\[ M = \ell^2 \left( \frac{1}{r} - \frac{3M}{r^2} \right) = \frac{\ell^2}{r^2} \left( r - 3M \right), \]

so
\[ \ell^2 = \frac{Mr^2}{r - 3M}. \quad \text{(H)} \]

From this
\[ 1 + \frac{\ell^2}{r^2} = 1 + \frac{M}{r - 3M} = \frac{r - 2M}{r - 3M} = \frac{r}{r - 3M} \left( 1 - \frac{2M}{r} \right), \]

summarize,
\[ \frac{\ell^2}{r^2} = \frac{r}{r - 3M} \left( 1 - \frac{2M}{r} \right). \quad \text{(I)} \]

Now put (I) into (F), and you get \( e^2 \) as a fu. of \( r \) for circular orbits,
\[ e^2 = \frac{r}{r - 3M} \left( 1 - \frac{2M}{r} \right)^2. \quad \text{(J)} \]
Now divide (i) by (H) and you get essentially (9.45),

\[
\frac{e^2}{l^2} = \frac{1}{M_r} \left(1 - \frac{2M}{r}\right)^2.
\]

This makes it easy to get (9.46).

The trick in all of this is don't try to solve for \( r \) as a fn. of \( \frac{M}{l} \). That's done in the book, Eq. (9.34), but the square root makes the formula messy so avoid using it.