fact for the student to ponder as gamma rays of this energy are almost completely attenuated) so the extrapolation is not too great.

The low energy range is easily measured with a Si(Li) detector. Since it is possible to use x rays for calibration, the extrapolation problem does not exist. An extremely low energy datum is available using a $^{241}$Am source with this detector. One of the Np L x rays falls on the edge, but a thin absorber, e.g., 0.3 mm of Cu, suppresses the x rays without attenuating the 59 keV gamma ray significantly and gives a clean Compton edge at 11 keV. Note that these two low energy points do little to constrain the value of $m_e c^2$ since from Eq. (7) the error in the rest energy is magnified for low $T$, i.e., $d(m_e c^2)/dT$ approaches infinity as $T$ approaches zero. Consequently, a measurement of the Compton edge for $^{241}$Am with an uncertainty of 0.1 keV yields an uncertainty greater than 5 keV in $m_e c^2$ and the effect is seen in Fig. 3. These low energy data do make the plots where $\beta$ is the abscissa much more convincing.

The error in the measurement of the Compton edge is about $\pm 1$ keV if the dispersion of the spectra is about 1 keV per channel and there are at least 300 counts per channel. It should be remembered that Compton scattering will be more important in a small detector than a large one which will have a higher probability of multiple scatterings. This means, however, that there is a low efficiency for high energy gamma rays. Most spectra can be taken in 5 to 15 min with source intensities of 1 to 10 $\mu$Ci although the high energy points often take longer.

V. CONCLUDING REMARK

The experiment described above is easy to carry out and analyze and provides a convincing experimental demonstration of the necessity of special relativity and a precise measurement of the electron rest mass. The theory only requires a modern physics course. The difficulty that so many of the interesting, and often displayed, relations, e.g., $p$ vs $\beta$, are flawed by the effects of correlated uncertainties should be pointed out to the student. We end our discussion with the suggestion that the student describe an experiment which does not suffer from this problem. The usual, and satisfactory, answer is to measure the velocity and momentum of a particle separately. For example, a velocity selector followed by the measurement of the radius of curvature in a magnetic field. Such an experiment is simpler in concept but more difficult in execution than the present experiment.

10. The program is available on request. Requests for direct transfer can be made to "jolliette@physics.hope.edu" on Internet.

Dispersion-free linear chains

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General formulas are given for the masses and spring constants of one-dimensional finite chains with linear dispersion relations, examples of which were given by Herrmann and Schmälzle in 1981 in their discussion of a well-known collision apparatus. The mathematical similarity to the problem of a Boson in a constant magnetic field is shown. The explicit formulas make a study of the continuum limit possible. This is shown to be related to the system of uniform rods studied by Bayman in 1976. Examples are given of chains with quadratic dispersion relations. Resonances that give singularities in the interaction time are discovered in certain chains of elastic spheres.

I. INTRODUCTION

Numerous papers$^1$ have been written on the ball collision apparatus shown in Fig. 1. When playing with such a device, people usually get the impression that if $n_m$ balls are drawn aside and released, $n_m$ balls will move away at the far end after the collision, the remaining balls being motionless (this occurrence will be called "perfect transmission" in this paper). There is also an impressive list$^2$ of books in which it is claimed that for the case of elastic balls, this outcome is determined by the laws of conserva-
limit of low velocities where elastic waves within the balls can be neglected.\textsuperscript{3}

The case of one ball incident with velocity \( v_0 \) on two balls of the same mass initially in contact and at rest is commonly discussed because it is the simplest case for which the conservation laws do not fully determine the outcome. The final velocities are\textsuperscript{4}

\[
\begin{align*}
b = 1: & \quad v_1/v_0 \approx -0.1303, \\
v_2/v_0 & \approx 0.1503, \quad v_3/v_0 \approx 0.9800 \\
b = 3/2: & \quad v_1/v_0 \approx -0.0710, \\
v_2/v_0 & \approx 0.0765, \quad v_3/v_0 \approx 0.9944.
\end{align*}
\]

These numbers do not depend on the mass, spring constant, or initial velocity. Agreement with experiment is good.

Then in 1981 Herrmann and Schmülling made a remarkable discovery. They found that certain chains of balls (gliders) of different masses interacting via various Hooke’s law forces do display the perfect transmission phenomenon. These chains have eigenfrequencies in the ratios of the first \( n \) integers (when the balls are in contact). The authors describe these chains as dispersion free. A “pulse” of any number of incident balls \( n_0 \) is unmodified after leaving the chain (perfect transmission), although during the interaction all of the balls are in motion. The chains they presented in their paper are reproduced here as entries 3 and 4 in Table I. They offered to send their method of calculating these numbers to the interested reader. The straightforward way to find these chains is to calculate the eigenfrequencies as functions of the masses and spring constants, and then to solve the complicated equations that result when setting the eigenfrequencies equal to the desired frequencies. It is therefore surprising that the solutions are such simple rational numbers.

In Sec. II of the present paper a general solution to these equations is presented. The mathematical relation to the problem of a quantum mechanical magnetic moment with integer spin in a constant magnetic field is shown. The general formulas make the study of the continuum limit possible (Sec. III). In Secs. IV and V, it is shown that perfect transmission occurs in other \( b=1 \) and \( b=3/2 \) chains. It is seen that certain \( b=3/2 \) chains (corresponding to ideal chains of spherical balls) have interaction times that can be made arbitrarily long by changing the masses slightly near values that give perfect transmission. The prospects of observing these “resonances” experimentally are discussed. The conclusion includes a discussion of the applicability of the term “dispersion free.”

### Table I. Masses and spring constants for Hooke’s law chains with eigenfrequencies proportional to the integers up to the number of springs. The normalization is such that \( c_j/c \) is unity and the lowest eigenfrequency is \( (c/m)^{1/2} \). In each row, the entries are \( m_0, c_1, m_2, \ldots, c_n, m_n \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>Masses over ( m ) and spring constants over ( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2 1 2</td>
</tr>
<tr>
<td>2</td>
<td>1 1 1</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} )</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} )</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} )</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
m_k &= \frac{2 \left( \frac{n}{2} \right)^2}{n \left( \frac{2n}{k} \right)} m, \quad k = 0, \ldots, n, \\
c_j &= (2j-1) \left( \frac{n-1}{2j-1} \right)^2 c, \quad j = 1, \ldots, n
\end{align*}
\]

II. A GENERAL SOLUTION FOR HOOKE’S LAW CHAINS WITH PERFECT TRANSMISSION

In this section the following result is proved. The chain of \( n+1 \) balls and \( n \) springs given by

\[
\begin{align*}
m_k &= \frac{2 \left( \frac{n}{2} \right)^2}{n \left( \frac{2n}{k} \right)} m, \quad k = 0, \ldots, n, \\
c_j &= (2j-1) \left( \frac{n-1}{2j-1} \right)^2 c, \quad j = 1, \ldots, n
\end{align*}
\]
Table II. Masses over \( m \) and spring constants over \( c \) for Hooke's law chains with eigenfrequencies proportional to the squares of the integers up to the number of springs. Since the chains are symmetric, only the left half is specified. The prime factorization of the numerator and denominator is given.

<table>
<thead>
<tr>
<th>( n )</th>
<th>Mass over ( m )</th>
<th>Spring constant over ( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{2 \cdot 11}{27} )</td>
<td>( \frac{2 \cdot 5 \cdot 7}{677} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{257}{2 \cdot 3^2 \cdot 19} )</td>
<td>( \frac{2 \cdot 19 \cdot 257}{5^2 \cdot 7 \cdot 13 \cdot 17} )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{2 \cdot 5 \cdot 7^2}{677} )</td>
<td>( \frac{2 \cdot 9 \cdot 7^2 \cdot 677}{13 \cdot 17 \cdot 29^2 \cdot 41} )</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{83 \cdot 109 \cdot 631}{3^4 \cdot 43 \cdot 2309} )</td>
<td>( \frac{2 \cdot 43 \cdot 83 \cdot 109 \cdot 631}{3 \cdot 5 \cdot 11 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 43 \cdot 61 \cdot 2309} )</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{5 \cdot 41(83 \cdot 109 \cdot 631)^2}{3^4 \cdot 11 \cdot 13 \cdot 17^3 \cdot 29 \cdot 37 \cdot 43 \cdot 61 \cdot 2309} )</td>
<td>( \frac{2 \cdot 5 \cdot 83 \cdot 109 \cdot 631^2}{3^5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 43 \cdot 61 \cdot 2309} )</td>
</tr>
</tbody>
</table>

has eigenfrequencies \( (c/m)^{1/2} \), \( 2(c/m)^{1/2} \), ..., \( n(c/m)^{1/2} \), and has the perfect transmission property. Here \( m \) is a mass constant, \( c \) is a spring constant (\( c=c_1=c_n \)), and \( (\xi) \) is the usual binomial coefficient. Note that \( m_k=m_{n-k} \) and \( c_j=c_{n+1-j} \). A few examples are shown in Table I.

We start with the equations of motion:

\[
\begin{align*}
\frac{d^2 x_0}{dt^2} &= -c_1 \cos(x_0-x_1), \\
\frac{d^2 x_k}{dt^2} &= c_k \cos(x_{k-1}-x_k) - c_{k+1} \cos(x_k-x_{k+1}), \\
\quad \text{for } k=1,\ldots,n-1, \\
\frac{d^2 x_n}{dt^2} &= c_n \cos(x_{n-1}-x_n).
\end{align*}
\]

(3)

In terms of the relative displacements \( \xi_k=x_{k-1}-x_k \), these are

\[
\frac{d^2 \xi_k}{dt^2} = -\left( \frac{c_1 + c_1}{m_0 + m_1} \right) \cos(\xi_1) + \frac{c_2}{m_1} \cos(\xi_2),
\]

\[
\frac{d^2 \xi_{n-1}}{dt^2} = -\left( \frac{c_{n-1} + c_n}{m_{n-1} + m_n} \right) \cos(\xi_n) + \frac{c_n}{m_{n-1}} \cos(\xi_{n-1}).
\]

(4)

Let us define the \( n \times n \) diagonal matrix \( C \) to have entries \( c_1, c_2, \ldots, c_n \), the \( n \times n \) tridiagonal matrix \( M \) to have \(-\frac{1}{m_0 + 1/m_1}, -\frac{1}{m_1 + 1/m_2}, \ldots, -\frac{1}{m_{n-1} + 1/m_n}\) on its main diagonal and \( 1/m_1, 1/m_2, \ldots, 1/m_{n-1} \) on the first subdiagonals, and the column vector \( \xi \) with components \( \xi_1, \xi_2, \ldots, \xi_n \). In terms of these quantities and for times when all of the \( \xi_k \) are non-negative, Eqs. (4) are

\[
\frac{d^2 \xi}{dt^2} = MC\xi.
\]

(5)

It is advantageous to work with a symmetric matrix, so we define the \( n \times n \) diagonal matrix \( D \) to have entries


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We define the column vector \(u = D \xi\). Left multiplication of the above equation by \(D\) gives (after using \(C = D^2\))

\[
d^2u \over dt^2 = DMDu. \tag{6}
\]

This is a system of \(n\) coupled differential equations.

We now consider the system of \(2n+1\) coupled differential equations

\[
d^2v \over dt^2 = -L_y^2 v, \tag{7}
\]

where \(L_y^2\) is the usual matrix representation (with respect to the basis of eigenstates of \(L_y^2\) of the operator for the \(y\) component of angular momentum for a total angular momentum quantum number equal to \(n\) \((\hbar = 1)\), and \(v\) is a column vector of \(2n+1\) functions of the dimensionless variable \(\tau\). For such \(2n+1\) component vectors we use the standard convention for angular momentum indices: \(m = n, n-1, \ldots, -n\). Since \(L_y^2\) only has nonzero elements on its main diagonal and second sub- and superdagonals \((4L_y^2 = 2L_x^2 - 2L_z^2 - \hat{L}_y^2 - \hat{L}_z^2)\), this system decouples into a set of \(n\) coupled different equations relating the functions

\[
(v_{i-1}, v_{i-2}, \ldots, v_{n-2}, v_{n-1}) \tag{8}
\]

and a set of \(n+1\) coupled differential equations for indices \(n, n-2, \ldots, -n\). A straightforward calculation shows that the set of equations for \((8)\) is the same as \((6)\) with \(t \rightarrow \tau (m/c)^{1/2}\), that is, that the matrix \(-m/cDMD\) is the same as the matrix obtained by taking \(L_y^2\) retaining only the elements with both indices among those indicated in \((8)\). Two relations [obtained directly from \((2)\)] useful in showing this are

\[
c_k = m_k = kc(n-k+1/2)/m, \tag{9}
\]

\[
c_k = m_{k-1} = c(k-1/2)(n-k+1)/m, \tag{10}
\]

but the details of the calculation will not be reproduced here.

An eigenvector \(w^{(k)}\) of \(L_y\) with eigenvalue \(k\) \((k = n, \ldots, -n)\) has components given by

\[
w^{(k)}_m = \frac{\Gamma m^{a_k} m}{\Gamma(n/2)} = \frac{(n+m)(n-m)}{(n+k)(n-k)} \tag{11}
\]

where in the notation of Rose

\[
d^2v/\over dt^2 = -L_y^2 v, \tag{12}
\]

\[
L_y^2 = L_x^2 - L_z^2 - \hat{L}_y^2, \tag{13}
\]

Any solution of this equation or of the equation corresponding to \(B \) field in the opposite direction will be a solution of \((7)\). For our purposes, it will be sufficient to consider the linear combination

\[
v = e^{-i\tau L_y^2} v_+, e^{-i\tau L_y^2} v_-, \tag{14}
\]

where \(v_+\) and \(v_-\) are arbitrary constant \(2n+1\) component column vectors, and the usual power series definition is used for the exponential function of a matrix (since we are only interested in certain particular solutions, it is not necessary to address the question of finding the general solution to \((7)\)). An equivalent solution is

\[
v = \sin (L_y^2) v_+ + \cos (L_y^2) v_-. \tag{15}
\]

We are only interested in the components of \(v\) indicated in \((8)\). The initial conditions for these components are those for \((6)\),

\[
u(0) = 0, \quad \frac{du_k}{dt}(0) = r \delta_{k, m_k} (m/c)^{1/2}, \tag{16}
\]

where \(r\) is defined to be \(v_0 (c \pi, m/c)^{1/2}\). These conditions describe the case of \(n_\infty\) balls incident with velocity \(v_0\). The relations \(v(\tau = 0) = v_\tau\) and \((dv/d\tau)(\tau = 0) = L_y v_\tau\) follow from \((15)\) and show that the initial conditions \((16)\) will be satisfied by taking the entire \(v_\tau\) vector to be zero and finding an appropriate \(v_\tau\).

The vector \(v_\tau\) must have the property that the components indicated in \((8)\) of \(L_y^2 v_\tau\) are \((0, \ldots, 0, \tau, 0, \ldots, 0)\) where the position of the \(\tau\) in this list is \(n_\infty\). \(L_y\) is not invertible; however, we may construct such a vector as follows.

We start by "truncating" \(w^{(0)}\). The \(2n+1\) component column vector \(f^{(n_\infty)}\) is defined by

\[
f^{(n_\infty)}_m = \begin{cases} \frac{w^{(0)}_m}{(m = n, n-1, \ldots, n+2 - 2n_\infty) \tag{17} & m = n+1 - 2n_\infty, \ldots, -n} \\
0 & (m = n+1 - 2n_\infty, \ldots, -n) \end{cases}
\]

Because of the structure of the \(L_y\) matrix, \(L_y f^{(n_\infty)}\) will only have one nonzero entry. This entry contains a factor of \((n+1-n_\infty)(n_\infty-1/2)^{1/2}\) from \(L_y\) and a factor of \(w^{(0)}_m \) from \(f^{(n_\infty)}_m\). It is located at position \(n+1-2n_\infty\). Thus we define

\[
v^{(n_\infty)}_\tau = \frac{r f^{(n_\infty)}_m}{(n+1-n_\infty)(n_\infty-1/2)^{1/2} w^{(0)}_{n+2 - 2n_\infty}}. \tag{18}
\]

This vector has the property that

\[
(n+1-n_\infty)(n_\infty-1/2)^{1/2} v^{(n_\infty)}_\tau = \delta_{m, n+1-2n_\infty}. \tag{19}
\]
Now the solution of Eq. (4) for the case of \( n_{\text{in}} \) balls incident with velocity \( v_0 \) may be written as

\[
\xi_k^{(n_{\text{in}})}(t) = \frac{1}{\sqrt{c_k}} \sin \left( L_p (c/m)^{1/2} t \right) \psi_s^{(n_{\text{in}})}_{n+1-2k}.
\] (20)

[In the discussion of the contact condition below, Eq. (20) is written out explicitly for two values of \( n_{\text{in}} \).] Note that the vector \( \psi_s^{(n_{\text{in}})} \) was not uniquely determined by condition (19). Any multiple of \( w^{(0)} \) could have been added to it. However, since the terms in the sine series above always contain at least one factor of \( L_p \), the \( \xi_k \) are uniquely determined.

Differentiating Eq. (20) with respect to time generates a factor of \( L_p (c/m)^{1/2} \) which may be combined with the \( \psi_s^{(n_{\text{in}})} \) vector to give a vector with only one nonzero entry (at position \( n+1-2n_{\text{in}} \)). This vector selects column \( n+1-2n_{\text{in}} \) of the \( \cos \left( L_p (c/m)^{1/2} t \right) \) matrix, and the final index in Eq. (20) indicates that entry \( n+1-2k \) of this vector is to be taken. This matrix element may be expressed in terms of the \( d \) matrix in Eq. (11) by writing the cosine as half the sum of two exponentials and using a symmetry of the \( d \) matrices to reduce this to a single term. The resulting velocities are

\[
\frac{d \xi_k^{(n_{\text{in}})}}{dt} (t) = v_0 \psi_s^{(n_{\text{in}})} \frac{(c/m)^{1/2} t}{2} \right)_{n+1-2k,n+1-2n_{\text{in}}}. \] (21)

Since \( \sin (\pi L_p \tau) \) is related to the sine of the diagonal matrix of eigenvalues of \( \pi L_p \) by a similarity transformation, \( \sin (\pi L_p \tau) \) is the \((2n+1) \times (2n+1)\) zero matrix. Therefore, Eq. (20) shows that at time \( t = \pi (m/c)^{1/2} \) all the \( \xi_k^{(n_{\text{in}})} \) are again zero. Equation (21) shows that at this time the relative velocities \( d \xi_k^{(n_{\text{in}})} / dt \) are

\[
0.
\]

There is no relative motion between neighboring balls, except between \( n-n_{\text{in}} \) and \( n+1-n_{\text{in}} \). This means that the last \( n_{\text{in}} \) balls move away with velocity \( v_0 \), leaving the others at rest.

It remains only to show that the equations of motion (5) are valid from time \( t=0 \) to time \( t = \pi (m/c)^{1/2} \), that is, to show that all of the \( \xi_k^{(n_{\text{in}})} \) are greater than or equal to zero throughout this time interval. For one ball incident, Eq. (20) gives

\[
\xi_k^{(n_{\text{in}}=1)} (t) = v_0 (m/c)^{1/2} \frac{2}{c_k} \frac{n-1}{k-1} \left[ \cos (\tau/2) \right]^{2n+1-2k} \times \left[ \sin (\tau/2) \right]^{2k-1} \] (22)

and for two balls incident, Eq. (20) gives a result which may be written as

\[
\xi_k^{(n_{\text{in}}=2)} (t) = \frac{2v_0 (m/c)^{1/2}}{c_k} \frac{n-1}{(k-1)} \times \left[ \cos (\tau/2) \right]^{2n-2k} \left[ \sin (\tau/2) \right]^{2k-3} \times \left[ \frac{(n-k)\sin (\tau/2) - (k-1)}{2n-1} \right] \times \frac{(k-1)(n-k)}{2n-1} \right]. \] (23)

From these expressions it is apparent that all the \( \xi_k (t) \) are greater than zero for \( 0 < t < \pi (m/c)^{1/2} \). The statement that this is true for arbitrary \( n_{\text{in}} \) can be shown to be equivalent to a statement about the much studied Jacobi polynomials.\(^7\)

If \( n_1, n_2, \) and \( n_3 \) are non-negative integers then

\[
\int_{x_0}^{1} P_{2n_1+1}^{(2n_2,2n_3)} (x) \frac{P_{2n_1+1}^{(2n_2,2n_3)} (x)}{1-x^2} w^{(n_1,n_2)} (x) dx > 0
\]

for \(-1 < x_0 < 1\).

Here \( P_{2n_1+1}^{(2n_2,2n_3)} (x) \) is a Jacobi polynomial and \( w^{(n_1,n_2)} (x) \) is the weight function for the Jacobi polynomials. The author is not aware of a proof of this statement, although he has verified it for the values of \( n_1, n_2, \) and \( n_3 \) necessary to show that all the \( \xi_k (t) \) are greater than zero for \( 0 < t < \pi (m/c)^{1/2} \) and arbitrary \( n_{\text{in}}, \) for \( n \) up to 20.

III. CONTINUUM LIMIT

We now consider the limit \( n \to \infty \), length = constant. In this limit the chain becomes a one-dimensional medium of varying linear mass density \( \rho \). Its behavior under compression can be described by a linear modulus of elasticity \( \gamma \) that is a function of position. However, if it is stretched there is no restoring force. If we let \( s \) be a coordinate that runs from \(-1\) at one end to \(+1\) at the other end, we get from Eq. (2)

\[
\rho (s) = \rho_0 (1-s^2)^{-1/2},
\]

\[
\gamma (s) = \gamma_0 (1-s^2)^{-1/2},
\]

where \( \rho_0 \) and \( \gamma_0 \) are constants. These expressions are obtained by using Stirling’s approximation

\[
(m \approx (2n+1)^{1/2} / n e^{-n}.
\]

First let us consider a medium that is similar to the one described above, the only difference being that the forces
are described by $\gamma(s)$ for both positive and negative strain. Let $y(s,t)$ be the displacement of the medium in the $+s$ direction at point $s$ and time $t$. Then $\frac{\partial y(s,t)}{\partial s}$ is the strain and $\gamma(s) \frac{\partial y(s,t)}{\partial s}$ is the tension. The equation of motion is

$$\rho(s) \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial s} \left( \gamma(s) \frac{\partial y}{\partial s} \right) - \gamma(s) \frac{\partial^2 y}{\partial s^2}$$  \hspace{1cm} (27)

and the boundary conditions are

$$\lim_{s \to \pm 1} \left( \gamma(s) \frac{\partial y(s,t)}{\partial s} \right) = 0.$$  \hspace{1cm} (28)

If $y(s,t) = Y(s) e^{-\sqrt{\nu}(\gamma_0/p_0)^{1/2} t}$ is an eigenmode then Eq. (27) gives

$$(1-s^2) Y'' - s Y' + \nu^2 Y = 0.$$  \hspace{1cm} (29)

This is the Chebyshev Equation\(^8\) (formerly transliterated as Tchebychev, Tchebycheff, or Tschebycheff). The solutions that satisfy condition (28) are the Chebyshev polynomials $T_n(s)$, $n=0, 1, 2, 3, ...$. In the limit $n \to \infty$, an eigenvector of $MC$ with eigenvalue $m$ becomes a continuous function of $s$. These are then proportional to $dT_m(s)/ds$ (since the $\xi_k$ were relative displacements). For example an eigenvector of $MC$ with eigenvalue 1 is $(1,1,1,...,1)$ and $dT_1(s)/ds = 1$.

The dispersion-free nature of this system is not very apparent in this form. Defining a new independent variable $\theta$ by $s = \cos \theta$, $-\pi < \theta < 0$, Eq. (27) becomes

$$\rho_0 \frac{\partial^2 y}{\partial \theta^2} = \gamma_0 \frac{\partial^2 y}{\partial s^2}$$  \hspace{1cm} (30)

which is the familiar wave equation of a uniform one-dimensional medium. As shown by Bayman,\(^9\) such systems do possess the perfect transmission property. If $n_{in}$ one-dimensional rods collide with some rods at rest and in contact, $n_{in}$ rods will leave the far end with the initial velocity after a time equal to the total length divided by the speed of sound in the rods. After the separation, there are no internal waves present in any of the rods. Only compressive waves are involved, so the equations of motion remain valid throughout the interaction. The same considerations apply to our system: Let $s_n$ be a real number between $-1$ and $+1$. Then if the section from $-1$ to $-s_n$ collides with the section from $-s_n$ to $+1$, the section from $+s_n$ to $+1$ will move away after the interaction, and there will be no internal excitations.

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Fig. 4. Plots of the dimensionless separation times $\theta_1$ and $\theta_2$ and of the final velocities over the incident velocity as functions of the mass ratio $\alpha$ for an elastic sphere interaction force. In order to identify the curves, the region between curve 1 and curve 2 is shaded when curve 2 is higher.
IV. OTHER HOOKE’S LAW CHAINS WITH PERFECT TRANSMISSION

The symmetry properties of the eigenmodes of the chains described in Sec. II can help to give us a more intuitive understanding of why these chains possess the perfect transmission property. The mode corresponding to frequency \( j (c/m)^{1/2} \) is symmetric or antisymmetric as \( j \) is odd or even. Since the \( d\xi/dt \) vector is a linear combination of these vectors times \( \cos[j (c/m)^{1/2} t] \), we see that at time \( t = \pi m/c \) the antisymmetric modes have the same coefficient as at time \( t = 0 \) while the coefficients of the symmetric modes have their signs changed. Thus

\[
\frac{d^n \xi_k}{dt^n} \left[ \pi (m/c)^{1/2} \right] = -\frac{d^n \xi_{n+1-k}}{dt^n} (0). \tag{31}
\]

Looking at this argument, we see that it would also apply if the frequencies were \( j^q (c/m)^{1/2} \), where \( q \) is a positive integer. This motivates us to look for chains with \( q = 2 \), for example. Such chains exist, and again the numbers involved are rational. The first six chains are shown in Table II. For lengths up to four balls, these chains have the perfect transmission property for all possible values of \( n_m \). However, for the longer chains, this is only true for \( n_m \) equal to 1 or \( n \); for other values of \( n_m \) the balls do not separate simultaneously.

V. A GENERAL LOOK AT THREE-BALL SYMMETRIC PERFECT TRANSMISSION CHAINS

What are all of the symmetric three-ball perfect transmission chains? There is only one parameter that needs to be varied: \( \alpha = m_0/m_1 = m_2/m_1 \). We may treat the cases of a Hooke’s law interaction (\( b = 1 \)) and spherical elastic ball interaction (\( b = 3/2 \)) simultaneously. The dimensionless equations of motion are

\[
\frac{d^2}{d\theta^2} (\xi_1) = \begin{pmatrix} -1 - 1/\alpha & 1 \\ 1 & -1 - 1/\alpha \end{pmatrix} \begin{pmatrix} \text{pos}(\xi_1) \\ \text{pos}(\xi_2) \end{pmatrix}, \tag{32}
\]

with initial conditions

\[
\xi_1 = \xi_2 = 0, \quad \frac{d\xi_1}{d\theta} = 1, \quad \frac{d\xi_2}{d\theta} = 0. \tag{33}
\]

A change in the spring constant or an overall multiplication of the masses by a number just rescales the time variable. For \( b = 3/2 \) a change in the initial velocity also rescales the time variable.

Let \( \theta_1 \) be the time when balls 1 and 2 separate. By this we mean the point in time at which the interaction between balls 1 and 2 ends (\( \xi_1 = 0 \)). Note that with this definition, balls with a relative separation of \( \xi = 0 \) are considered separate. Similarly, let \( \theta_2 \) be the time when balls 2 and 3 separate. Figure 2 shows a plot of \( \theta_1 \) and \( \theta_2 \) and the final relative velocities as a function of \( \alpha \) for \( b = 1 \). The plot shows three cases of perfect transmission. The first one (at \( \alpha = 3/2 \)) corresponds to the three-ball chain of Sec. II. The second one (at \( \alpha = 15/2 \)) corresponds to the three-ball \( q = 2 \) chain of the previous section. [A simple calculation shows that these chains have \( \alpha = (2^{2q}-1)/2 \). The third one (at \( \alpha = 35/2 \)) corresponds to a three-ball chain with \( \omega_2/\omega_1 = 6 \); the argument of Sec. III applies to this case, too.

We also see that there are cases in which balls 2 and 3 move away together after the interaction. The ones shown at \( \alpha = 4 \) and \( \alpha = 12 \) correspond to \( \omega_2/\omega_1 = 3 \) and \( \omega_2/\omega_1 = 5 \), respectively. An argument similar to that of Sec. III explains these.

These various cases could be implemented with gliders on an air track, as a classroom demonstration. The middle glider would have to be rather light weight though.

Figure 3 is an aid to explaining the kinks in the \( \theta_1 \) and \( \theta_2 \) curves in Fig. 2. Figure 3(b) shows the relative positions as functions of time for a value of \( \alpha \) slightly to the left of
\( \alpha = 3/2 \). Balls 1 and 2 separate before balls 2 and 3. The relative velocity between balls 2 and 3 is large and negative so they separate soon thereafter. Figure 3(a) is for a value of \( \alpha \) slightly to the right of \( \alpha = 3/2 \). Balls 2 and 3 separate first. However, \( d\xi_1/d\theta \) is almost zero at this time. Since the equation of motion for \( \xi_1 \) is now that of a simple harmonic oscillator, it takes about a quarter-period of this simple harmonic oscillator for \( \xi_1 \) to reach 0, regardless of how small \( \xi_1 \) was at time \( \theta_2 \).

Figure 4 is the same as Fig. 2, but for the case \( b = 3/2 \). The behavior is similar to that discussed above except the jumps are now singularities. The reason is that in a non-dimensional potential the time \( \xi = 0 \) if released from a positive \( \xi_0 \) at rest depends on \( \xi_0 \), becoming infinite in the limit of small \( \xi_0 \).

For values of \( \alpha \) slightly to the right of a point of perfect transmission, balls 1 and 2 are slightly compressed and have almost no relative motion when ball 3 leaves. It takes a long time for them to separate. Otherwise the situation is similar to that discussed concerning Fig. 4. An example is shown in Fig. 5.

These one-sided resonances (especially the one at \( \alpha \approx 1.2019 \)) could be observed. Measurement of the interaction time of metal ball collisions is normally done by using the contact between the balls to close an electric circuit. Either the middle ball would have to be modified so that its mass could be tuned in a range around 83% of the original value, or the first and last would have to have their masses increased (this could be done without drilling, e.g., by gluing additional mass to two diametrically opposed points).

VI. CONCLUSIONS

We have seen perfect transmission in a variety of chains, including chains with a \( b = 3/2 \) interaction for which the superposition principle is not valid. The continuum limit of the chains discussed in Sec. II was seen to be related to a dispersion-free wave equation by a simple change of variables. To extend the notion of dispersion-free behavior to finite chains, we could use the perfect transmission property as a requirement. However the \( q > 1 \) chains of Sec. IV show that this does not necessitate a linear relation between frequency and mode number as implied in Herrmann and Schmälzle's paper.

In Sec. V we saw some surprising behavior in three-ball chains, suitable for classroom demonstrations. An experimental demonstration of the resonances in Fig. 4 would be operating in a regime in which internal waves and imperfect coefficients of restitution are important.

\(^{1}\)The following papers are all from the American Journal of Physics.

\(^{2}\)See the citations in Ref. 1.


\(^{4}\)See Chapman in Ref. 1.


\(^{9}\)See Bayman in Ref. 1.