Inferring Planetary Obliquity Using Rotational & Orbital Photometry

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ABSTRACT

The obliquity of a terrestrial planet is an important clue about its formation and critical to its climate. Previous studies using simulated photometry of Earth show that continuous observations over most of a planet’s orbit can be inverted to infer obliquity. We extend this approach to single-epoch observations for planets with arbitrary albedo maps. For diffuse reflection, the flux seen by a distant observer is the product of the planet’s albedo map, the host star’s illumination, and the observer’s visibility of different planet regions. It is useful to treat the product of illumination and visibility as the kernel of a convolution; this kernel is unimodal and symmetric. For planets with unknown obliquity, the kernel is not known a priori, but could be inferred by fitting a rotational light curve. We analyze this kernel under different viewing geometries, finding it well described by its longitudinal width and latitudinal position. We use Monte Carlo simulation to estimate uncertainties on these kernel characteristics from variations in a planet’s apparent albedo. We demonstrate that the kernel properties are functions of obliquity and axial orientation, which may both be inferred even if planets are A) East-West uniform or spinning rapidly, or B) North-South uniform. We consider degeneracies in these inferences with a case study, and describe how to tell prograde from retrograde rotation for inclined, oblique planets. This approach could be used to estimate obliquities of terrestrial planets with modest time investment from flagship direct-imaging missions.

Key words: methods: analytical – planets and satellites: fundamental parameters – methods: statistical.

1 INTRODUCTION

The obliquity of a terrestrial planet encodes information about different processes. First, a planet’s axial alignment and spin rate inform its formation scenario. Numerical simulations have shown that the spin rates of Earth and Mars are likely caused by a few planetesimal impacts (Dones & Tremaine 1993), while perfect accretion produces an obliquity distribution that is isotropic (e.g. Kokubo & Ida 2007; Miguel & Brunini 2010). Conversely, Schlichting & Sari (2007) describe how prograde rotation is preferred to retrograde for a formation model with semi-collisional accretion.

Second, obliquity is important in controlling planetary climate. This has been studied in-depth for Earth under many conditions (e.g. Laskar et al. 2004; Pierrehumbert 2010), and high axial tilts can make planets at large semi-major axes more habitable (Williams & Kasting 1997). Furthermore, while the Earth’s spin axis is stabilized by the Moon (Laskar et al. 1993), obliquities of several Solar System bodies evolve chaotically (Laskar 1994). This influences searches for hospitable planets, as Spiegel et al. (2009) note that the habitability of terrestrial worlds may depend sensitively on how stable the climate is in the short-term.

A planet’s average insolation is set by stellar luminosity and semi-major axis; insolation at different latitudes is determined by obliquity and (for eccentric orbits) the axial orientation. Non-oblique planets have a warmer equator and colder poles which do not vary much throughout the year. Modest obliquities produce seasons at mid-latitudes because the sub-stellar point moves North and South during the orbit (Pierrehumbert 2010). Planets tilted $\geq 54^\circ$ receive more overall radiation near their poles and have large orbital variations in temperature (Williams & Pollard 2003). Thus, even limited knowledge of a planet’s obliquity can help constrain the spatial dependence of insolation and temperature.

Reflected light from terrestrial exoplanets will be studied with forthcoming optical and near-infrared space missions, such
as ATLAST/HDST (Kouveliotou et al. 2014) and LUVOIR (Postman et al. 2009). Time-resolved measurements of a rotating planet in one photometric band can reveal its rotation rate (Ford et al. 2001; Pallé et al. 2008; Oakley & Cash 2009); this helps determine Coriolis forces and predict large-scale circulation. Multi-band photometry can reveal colors of clouds and surface features (Ford et al. 2001; Fujii et al. 2010, 2011; Cowan & Strait 2013), and enables a longitudinal albedo map to be inferred from disk-integrated light (Cowan et al. 2009). High-cadence, reflected light measurements spanning a full planetary orbit constrain a planet’s obliquity and two-dimensional albedo map (Kawahara & Fujii 2010, 2011; Fujii & Kawahara 2012). This is appropriate for planets that orbit quickly, but full retrievals grow infeasible as orbital periods increase.

Other methods have been proposed for measuring planetary obliquities. Seager & Hui (2002) and Barnes & Fortney (2003) demonstrated constraints on oblateness and obliquity using ingress/egress differences in transit light curves; Carter & Winn (2010) extended and applied these techniques to observations of HD 189733b. Kawahara (2012) derived constraints on obliquity from modulation of a planet’s radial velocity during orbit, while Nikolov & Sainsbury-Martinez (2015) examined the Rossiter-McLaughlin effect at secondary eclipse for transiting exoplanets. One could also measure obliquity with infrared wavelengths, using polarized rotational light curves (De Kok et al. 2011) and orbital variations (e.g. Gaidos & Williams 2004; Cowan et al. 2013). We look to constrain a planet’s spin axis with diffuse reflected light photometry—an extension of Kawahara & Fujii (2010, 2011) and Fujii & Kawahara (2012) to observations at only one or two orbital phases.

Light curves of planets encode the viewing geometry and hence a planet’s obliquity because different latitudes are impinged by starlight at different orbital phases. To see this, consider a planet with no obliquity in an edge-on, circular orbit. The star always illuminates the Northern and Southern hemispheres equally, and we never view some latitudes more than others. If instead this planet were tilted, the Northern hemisphere would be lit first, then the Southern hemisphere half an orbit later. If the planet’s Northern and Southern hemispheres have different albedo markings, its apparent albedo (Qui et al. 2003; Cowan et al. 2009) would change during its orbit (left panel of Figure 1) and these changes can help one constrain the obliquity.

We may also learn about a planet’s obliquity as it rotates. Imagine a zero obliquity planet in a face-on, circular orbit: the observer always sees the Northern pole with half the longitudes illuminated. For an oblique planet, however, more longitudes would be lit when the visible pole leans towards the star, and vice versa. Zero obliquity planets in edge-on orbits are similar, since more longitudes are lit near superior conjunction, or fullest phase. If the planet has East-West albedo variations, then this longitudinal width will modulate the apparent albedo of the planet as it spins (right panel of Figure 1), which again helps one constrain the obliquity.

Our work is organized as follows: in Section 2 we summarize the observer viewing geometry and explain the reflective kernel, both in two- and one-dimensional forms. Section 3.1 introduces a case study planet and describes the kernel at single orbital phases; we consider time evolution in Section 3.2. We discuss our assumptions about determining the kernel in Section 4.1, then develop our case study in Sections 4.2 and 4.3, showing that we can infer obliquity using the kernel’s longitudinal width and latitudinal position. In Section 4.4 we discuss how to distinguish a planet’s rotational direction by monitoring its apparent albedo. Section 5 summarizes our conclusions. For interested readers, a full mathematical description of the illumination and viewing geometry is presented in Appendix A. Details about the kernel and its relation to a planet’s apparent albedo are described in Appendix B.

2 REFLECTED LIGHT

2.1 Geometry & Flux

The locations on a planet that contribute to the disk-integrated reflected light depend only on the sub-observer and sub-stellar positions, which both vary in time. A complete development of this viewing geometry is provided in Appendix A, which we summarize here. We neglect axial precession and consider planets on circular

Figure 1. The left panel shows apparent albedo as a function of orbital phase for an arbitrary planet with North-South and East-West albedo markings, seen edge-on with zero obliquity (black) or 45° obliquity (green). The average albedo of the green planet increases during the orbit because brighter latitudes become visible and illuminated; this does not happen for the black planet. The vertical bands are each roughly two planetary days, enlarged at right, where lighter shades represent the fuller phase. For clarity, the rotational curves are shifted and the zero obliquity planet is denoted by a dashed line. The apparent albedo of either planet varies more over a day when a narrower range of longitudes and albedo markings are visible and illuminated, and vice versa. Orbital and/or rotational changes in apparent albedo can help one infer a planet’s obliquity.
orbits. Assuming a static albedo map, the reflected light seen by an observer is determined by the colatitude and longitude of the sub-stellar and sub-observer points, explicitly $\theta_o$, $\phi_o$, $\theta_s$, and $\phi_s$. The intrinsic parameters of the system are the orbital and rotational angular frequencies, $\omega_{\text{orb}}$ and $\omega_{\text{rot}}$ (where positive $\omega_{\text{rot}}$ is prograde), and the planetary obliquity, $\Theta \in [0, \pi/2]$. Extrinsic parameters differ from one observer to the next: the orbital inclination, $i$ (where $i = 90^\circ$ is edge-on), and solstice phase, $\xi$ (the orbital phase of Summer solstice for the Northern hemisphere). We also define initial conditions for orbital phase, $\xi_0$, and the sub-observer longitude, $\phi_o(0)$. Reflected light is then completely specified by these seven parameters and the planet’s albedo map.

We consider only diffuse (Lambertian) reflection in our analysis. Specular reflection, or glint, can be useful for detecting oceans (Williams & Gaidos 2008; Robinson et al. 2010, 2014), but is a localized feature and a minor fraction of the reflected light at gibbous phases. The reflected flux measured by a distant observer is therefore a convolution of the two-dimensional kernel (or weight function, Fujii & Kawahara 2012), $K(\theta, \phi, S)$, and the planet’s albedo map, $A(\theta, \phi)$:

$$F(t) = \int K(\theta, \phi, S) A(\theta, \phi) \, d\Omega,$$

where $F$ is the observed flux, $\theta$ and $\phi$ are colatitude and longitude, and $S \equiv \{\theta_s, \phi_s, \theta_o, \phi_o\}$ implicitly contains the time-dependencies in the sub-stellar and sub-observer locations, described in Appendix A. For single-epoch observations of a planet with albedo markings, one would fit $F(t)$ to infer $A(\theta, \phi)$ and $K(\theta, \phi, S)$ at that orbital phase (similar to Cowan et al. 2009; Kawahara & Fujii 2010, 2011; Fujii & Kawahara 2012). This kernel could then constrain the planet’s obliquity. We therefore focus on $K(\theta, \phi, S)$ from Equation 1, which we can analyze independent of the albedo map.

2.2 Kernel

The kernel combines illumination and visibility, defined for diffuse reflection in Cowan et al. (2013) as

$$K(\theta, \phi, S) = \frac{1}{\pi} V(\theta, \phi, \theta_o, \phi_o) I(\theta, \phi, \theta_s, \phi_s),$$

where $V(\theta, \phi, \theta_o, \phi_o)$ is the visibility and $I(\theta, \phi, \theta_s, \phi_s)$ is the illumination. Visibility and illumination are each non-zero over one hemisphere at any time, and can be expressed as (Cowan et al. 2013):

$$V(\theta, \phi, \theta_o, \phi_o) = \max \left[ \sin \theta \sin \theta_o \cos (\phi - \phi_o) + \cos \theta \cos \theta_o, 0 \right],$$

$$I(\theta, \phi, \theta_s, \phi_s) = \max \left[ \sin \theta \sin \theta_s \cos (\phi - \phi_s) + \cos \theta \cos \theta_s, 0 \right].$$

Crucially, $S = f(G, \omega_{\text{rot}})$ where $G \equiv \{\xi(t), i, \Theta, \xi\}$ is the observer’s viewing geometry and $\xi(t)$ is orbital phase, as described in Appendix A. We can therefore express the kernel as

$$K(\theta, \phi, S) = K(\theta, \phi, G, \omega_{\text{rot}}),$$

though we will drop the rotational dependence for now because it does not affect our analysis. We return to rotational frequency in Section 4.4.

The non-zero portion of the kernel is a lune: the illuminated region of the planet that is visible to a given observer. The size of this lune depends on orbital phase, or the angle between the sub-observer and sub-stellar points. A sample kernel is shown at the top of Figure 2, where the purple and yellow contours are visibility and illumination, respectively. The peak of the kernel is marked with an orange diamond.

We begin by calculating time-dependent sines and cosines of the sub-observer and sub-stellar angles for a viewing geometry of interest (Appendix A). These are substituted into Equations 3 and 4 to determine visibility and illumination at any orbital phase. The two-dimensional kernel is then calculated on a 101 × 201 grid in colatitude and longitude.

2.3 Longitudinal Width

The two-dimensional kernel, $K(\theta, \phi, G)$, is a function of latitude and longitude that varies with time and viewing geometry. For observations with minimal orbital coverage or planets that are North-South symmetric, different latitudes are hard to distinguish (Cowan et al. 2013) and we use the longitudinal form of the kernel, $K(\phi, G)$, given by

$$K(\phi, G) = \int_0^{\pi} K(\theta, \phi, G) \sin \theta d\theta.$$

We can approximately describe $K(\phi, G)$ by a longitudinal mean, $\bar{\phi}$, and width, $\sigma_\phi$. These are defined in Appendix B1; examples are shown as vertical red lines in the bottom panel of Figure 2.

For any geometry, we can calculate the two-dimensional kernel and the corresponding longitudinal width. The mean longitude is unimportant by itself because, for now, we are only concerned with the size of the kernel. We compute a four-dimensional grid of kernel widths with $5^\circ$ resolution in orbital
phase (time), inclination, obliquity, and solstice phase. The result is $\sigma_\phi(\xi(t), i, \Theta, \xi_\phi) \equiv \sigma_\phi(G)$, and our numerical grid has size $73 \times 19 \times 19 \times 73$ in the respective parameters. Example contours from this array at first quarter phase, or $\xi(t) = 90^\circ$, are shown in the left panels of Figure 3. In these plots obliquity is radial: the center is $\Theta = 0^\circ$ and the edge is $\Theta = 90^\circ$. The azimuthal angle gives the orientation (solstice phase) of the planet’s obliquity.

### 2.4 Dominant Colatitude

For a given planet and observer, the sub-observer colatitude is fixed but the sub-stellar point moves North and South throughout the orbit if the planet has non-zero obliquity. This means different orbital phases will probe different latitudes, as dictated by the kernel. To analyze these variations we use the latitudinal form of the kernel, $K(\theta, G)$, explicitly:

$$K(\theta, G) = \int_{-\pi}^{\pi} K(\theta, \phi, G) d\phi. \quad (7)$$

We may describe $K(\theta, G)$ by its dominant colatitude (Cowan et al. 2012), $\bar{\theta}$, also defined in Appendix B1 and shown as a horizontal blue line at the bottom of Figure 2. We produce a four-dimensional dominant colatitude array, $\bar{\theta}(\xi(t), i, \Theta, \xi_\phi) \equiv \bar{\theta}(G)$, similarly to $\sigma_\phi(G)$ from Section 2.3. Sample contours from this array at first quarter phase are shown in the right panels of Figure 3.

### 3 KERNEL BEHAVIOR

The kernel characteristics—it’s longitudinal width and dominant colatitude—vary with the viewing geometry of the observer. We now consider how these characteristics encode the spin axis of a planet. As a case study, we will define the hypothetical planet $\Psi$ and its inclination:

$$\text{planet } \Psi \equiv \{i = 60^\circ; \Theta = 55^\circ; \xi_\phi = 260^\circ\}. \quad (8)$$

### 3.1 Phases

Considering a single orbital phase defines a three-dimensional slice through $\sigma_\phi(G)$ and $\bar{\theta}(G)$ that describes the kernel at that specific time. We show the longitudinal form of the kernel for planet $\Psi$ at different phases in the left panel of Figure 4. Lighter colors are fuller phases, indicating the kernel narrows as this planet orbits towards inferior conjunction, or $\xi(t) = 180^\circ$. The kernel width influences the rotational light curve at a given phase: narrower kernels can have larger amplitude variability in apparent albedo on a shorter timescale (e.g. right of Figure 1).

The latitudinal kernel for $\Psi$ is shown similarly in the right panel of Figure 4. We see that the kernel preferentially probes low and mid-latitudes during the first half-orbit. The dominant colatitude of $\Psi$, indicated by circles, also fluctuates during this portion of the orbit—and eventually shifts well into the Northern hemisphere after inferior conjunction (not shown). Note that the dominant colatitude is not always at the peak of the latitudinal kernel (see also Figure B1). Inferring a single dominant colatitude is difficult because we need flux from multiple phases to sense latitudinal variations at all. It is more realistic to infer an absolute value change in dominant colatitude from one phase to the next, $|\Delta \bar{\theta}|$, for planets with North-South asymmetry. Larger changes in dominant colatitude can make the apparent albedo vary more between orbital phases (e.g. left of Figure 1).
3.2 Time Evolution

Kernel width and dominant colatitude both vary throughout a planet’s orbit. We investigate this by slicing $\sigma_\phi(\xi)$ and $\bar{\psi}(\xi)$ along inclination, obliquity, and/or solstice phase. To start, we vary planet $\Psi$’s obliquity and track kernel width as shown in the left panel of Figure 5. The actual $\Psi$ is denoted by a dashed green line: this planet has a narrow kernel width during the first half-orbit that widens sharply after inferior conjunction. The overlapping traces of different obliquities make it hard to infer $\Psi$’s tilt from observations at only one phase.

We also show tracks of dominant colatitude in the right panel of Figure 5. None overlap until inferior conjunction, when the variation between all obliquities decreases. Planet $\Psi$ is again the dashed green line, and near the middle of all the tracks more often than for kernel width. If one already has some knowledge of the viewing geometry, then Figure 5 indicates at which phases the kernel best separates obliquities for $\Psi (\xi(t) \approx 120^\circ$, in this case).

We can instead vary the solstice phase of $\Psi$ while keeping its obliquity fixed (not shown). In most cases, solstice phase impacts the kernel width and dominant colatitude as much as the axial tilt. This is expected, since obliquity is a vector quantity with both magnitude and orientation.

4 DISCUSSION

4.1 Measurements

Before continuing in more detail, we must address our assumptions about determining the kernel. As noted in Section 2.1, one will not measure the kernel directly, but rather infer it by fitting a rotational light curve (following Cowan et al. 2009; Kawahara & Fujii 2010, 2011; Fujii & Kawahara 2012). The planetary inclination and orbital phase of observation must therefore be known to model the light curve accurately. Both angles might be obtained with a mixture of astrometry on the host star (e.g. SIM PlanetQuest, Unwin et al. 2008), direct-imaging astrometry (Bryden 2015), and/or radial velocity. We will demonstrate constraints on obliquity using the kernel by assuming inclination and orbital phase have each been measured with $10^\circ$ uncertainty.

Moreover, extracting the longitudinal width and dominant colatitude could be difficult in practice. Planets with completely uniform albedo are obviously not amenable to these methods. Changes in dominant colatitude can only be detected for planets with North-South albedo inhomogeneities. Conversely, one cannot extract the longitudinal width for planets lacking East-West albedo markings. Even if a planet has suitable albedo asymmetries, photometric uncertainty adds noise to the reflected light measured. Contrast ratios $\lesssim 10^{-11}$ are needed to resolve rotational light curves of an Earth-like exoplanet (Pallé et al. 2008), which should be achievable by a TPF-type mission with high-contrast coronagraph or starshade (Ford et al. 2001; Trauger & Traub 2007; Turnbull et al. 2012; Cheng-Chao et al. 2015). We will assume kernel widths and changes in dominant colatitude have been measured with $10^\circ$ and $20^\circ$ uncertainties apiece, as explained in Appendix B2.

Planetary radii are likewise important, since without them one cannot convert fluxes into apparent albedos (Qui et al. 2003; Cowan et al. 2009). Radii will likely be unknown, but could be approximated using mass-radius relations and mass estimates from astrometry or radial velocity, or inferred from bolometric flux using thermal infrared direct-imaging (e.g. TPF-I, Beichman et al. 1999; Lawson et al. 2008). Real planets may also have variable albedo maps, short-term from changing clouds and less so long-term from seasonal changes (Robinson et al. 2010), that could influence the apparent albedo on orbital timescales. These are difficulties that will be mitigated with each iteration of photometric detectors and theoretical models.

4.2 Longitudinal Constraints

For planets with East-West contrast in albedo, one can infer the kernel width at a given phase by fitting a rotational light curve. We can estimate the uncertainty on this width—from rotational variations in apparent albedo—by using Monte Carlo simulation (Appendix B2). The upper row of Figure 6 shows confidence regions for the obliquity and solstice phase of planet $\Psi$, using two example kernel widths: $25.2 \pm 10.0^\circ$ at $\xi(t) = 120^\circ$, and
32.8 ± 10.0° at ξ(τ) = 300°. We use a normalized Gaussian probability density for each width, and include Gaussian weights due to uncertainties on inclination and orbital phase, described in Section 4.1.

The true Ψ spin axis is shown in green, and is consistent with both isolated observations. Kernel width estimates at two orbital phases still allow all obliquities at 1σ, but exclude nearly 19 per cent of spin axes at 3σ. We obtain similar results for other orbital phases and planet parameters. These examples suggest that obliquity can be constrained for planets with variable albedo maps by extracting the kernel width, provided the variations are on timescales longer than the rotational period.

4.3 Latitudinal & Joint Constraints

With North-South contrast in albedo, one can fit two or more rotational light curves at different phases to infer the kernel’s change in dominant colatitude. We can estimate the uncertainty on this change—from orbital variations in apparent albedo—with further Monte Carlo simulation (Appendix B2). We reapply the probability density from Section 4.2 to the example absolute value change in dominant colatitude, 48.7 ± 20.0°, for planet Ψ between ξ(τ) = {120°, 300°}. This constraint is shown as blue regions at the lower left of Figure 6. The true Ψ spin axis is again inside the 1σ interval. The distribution is bimodal because we do not know the sign of the variation, meaning whether we are probing more Northern or Southern latitudes at the later phase.

We infer more with the product distribution of longitudinal and latitudinal constraints, shown as purple regions at the lower right of Figure 6. This is still bimodal, but the true spin axis is in a high-probability region and obliquities at either extreme can be excluded at 1σ. A distant observer will know that this planet’s obliquity has probably not been eroded by tides (Heller et al., 2011), and that the planet likely experiences obliquity seasons.

4.4 Pro/Retrograde Rotation

The sign of rotational angular frequency (positive = prograde) does not influence the kernel width or dominant colatitude at a given phase. Since σφ(G) and θ(G) are identical for retrograde rotation, the kernel characteristics will not distinguish prograde and retrograde planets. There is a formal degeneracy for edge-on, zero-obliquity cases: prograde planets with East-oriented maps have identical light curves to retrograde planets with West-oriented maps. The path of the kernel peak over either planet is the same, implying the retrograde rotation in an inertial frame is slower (Appendix B3). We show this scenario in the left panel of Figure 7, where the dashed brown line is the difference in prograde and retrograde apparent albedo. The orange and black planets are always equally bright because the same map features, in the upper panels, are seen at the same times.

However, the spin direction of oblique planets and/or those on inclined orbits may be deduced. Inclinations that are not edge-on most strongly alter a planet’s light curve near inferior conjunction, seen in the center panel of Figure 7: this planet’s properties are intermediate between the edge-on, zero obliquity planet and planet Ψ. While a typical observatory’s inner working angle would hide some of the signal, differences of order 0.1 in the apparent albedo would be detectable at extreme crescent phases. Alternatively, higher obliquity causes deviations that—depending on solstice phase—can arise around one or both quarter phases. This happens for Ψ in the right panel of Figure 7, where both effects combine to distinguish the spin direction at most phases.

Inclination and obliquity influence apparent albedo because the longitudinal motion of the kernel peak is not the same at all latitudes. One can break this spin degeneracy in principle, but we have not fully explored the pro/retrograde parameter space. In general, the less inclined and/or oblique a planet is, the more favorable crescent phases are for determining its spin direction.
5 CONCLUSIONS

We have performed numerical experiments to study the problem of inferring a planet’s obliquity based solely on time-resolved photometry. We have demonstrated that a planet’s obliquity will influence its light curve in two distinct ways, and that we can analyze the kernel of reflection independent of the planet’s albedo map. This kernel—the combination of stellar illumination and observer visibility—has a peak, a longitudinal width, and a dominant colatitude that vary in time, are functions of viewing geometry, and encode the planet’s obliquity. As long as a planet is not completely uniform, one can infer properties of both the kernel and albedo map. These characteristics help one determine the planet’s spin direction and constrain both components of its spin axis, including for maps that are East-West uniform (e.g. Jupiter-like) or North-South uniform (e.g. beach ball-like). Curiously, we find that kernel width is more useful in general, meaning obliquity is typically easier to infer from rotational rather than orbital information.

Moreover, one only needs to monitor a planet at a limited number of epochs to determine its obliquity. In our case study, we used observations at just two distinct phases to reduce the possible spin axes of planet Ψ by about 75 per cent at 1σ. Additional observations of order a week—rather than many months—could well constrain the true components, good news for inferring the obliquity of terrestrial exoplanets during direct-imaging surveys. We envision a triage approach for such missions: planets that vary in brightness the most, and thus have the easiest kernels to extract, will be the first for follow-up observations.

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Figure 7. Apparent albedo as a function of orbital phase, shown for an edge-on, zero obliquity planet at left, planet $\Psi$ at right, and an intermediate planet in the center. The black and orange curves correspond to prograde and retrograde rotation, respectively. The differences in apparent albedo are the dashed brown lines. A low rotational frequency is used for clarity; inferior conjunction occurs at $\xi(t) = 180^\circ$. The albedo maps are color-coded at top, where arrows indicate spin direction and the prime meridians are centered. Note that these maps are East-West reflections of each other. The edge-on, zero-obliquity curves are identical, while the curves for the intermediate planet and $\Psi$ grow more distinct. Edge-on, zero-obliquity planets are hopeless, but one can distinguish pro/retrograde rotation for inclined, oblique planets by monitoring their brightness, particularly near crescent phases.
APPENDIX A: VIEWING GEOMETRY

A1 General Observer

The time-dependence of the kernel is contained in the sub-observer and sub-stellar angles: \( \theta_s, \phi_s, \phi_s, \phi_s \). Since they do not depend on planetary latitude or longitude, these four angles may be factored out of the kernel integrals. However, the light curves are still functions of time, so we derive the relevant dependencies here.

In particular, we compute the sub-stellar and sub-observer locations for planets on circular orbits using seven parameters. Three are intrinsic to the system: rotational angular frequency, \( \omega_{\text{rot}} \in (-\infty, \infty) \), orbital angular frequency, \( \omega_{\text{orb}} \in (0, \infty) \), and obliquity, \( \Theta \in [0, \pi/2) \). Rotational frequency is measured in an inertial frame, where positive values are prograde and negative denotes retrograde rotation (for comparison, the rotational frequency of Earth is \( \omega_{\text{rot}} \approx 2\pi/23.93 \text{ hr}^{-1} \)). Two more parameters are extrinsic and differ for each observer: orbital inclination, \( i \in [0, \pi/2] \), and the initial sub-observer longitude, \( \phi_s(0) \in [0, 2\pi] \). These parameters are illustrated in Figure A1; other combinations are possible.

We define the orbital phase of the planet as \( \xi(t) = \omega_{\text{orb}} t + \xi_0 \). Without loss of generality we may set the first initial condition as \( \xi_0 = 0 \), which puts the planet at superior conjunction when \( t = 0 \). With no precession the sub-observer colatitude is constant,

\[
\theta_s(t) = \theta_s. \tag{A1}
\]

This angle can be expressed in terms of the inclination, obliquity, and solstice phase using the spherical law of cosines (bottom of Figure A1):

\[
\cos \theta_s = \cos i \cos \Theta + \sin i \sin \Theta \cos \xi, \quad \sin \theta_s = \sqrt{1 - \cos^2 \theta_s}. \tag{A2}
\]

The sub-observer longitude decreases linearly with time for prograde rotation, as we define longitude increasing to the East:

\[
\phi_s(t) = -\omega_{\text{orb}} t + \phi_s(0). \tag{A3}
\]

The prime meridian (\( \phi_s = \phi = 0 \)) is a free parameter, which we define to run from the planet’s North pole to the sub-observer point at \( t = 0 \). This sets the second initial condition, namely \( \phi_s(0) = 0 \), and means

\[
\cos \phi_s = -\omega_{\text{orb}} t, \quad \sin \phi_s = \sqrt{1 - \cos^2 \phi_s}. \tag{A5}
\]

Hence, the time evolution of the sub-observer point is specified by its colatitude and the rotational angular frequency.

The sub-stellar position is more complex for planets with non-zero obliquity. Consider an inertial Cartesian frame centered on the host star with fixed axes as follows: the \( z \)-axis is along the orbital angular frequency, \( \hat{z} = \hat{\omega}_{\text{orb}} \), while the \( x \)-axis points towards superior conjunction. The \( y \)-axis is then orthogonal to this plane using \( \hat{y} = \hat{z} \times \hat{x} \) (bottom of Figure A1). In these inertial coordinates, the unit vector from the planet center towards the host star is \( \hat{r}_p = -\cos \xi \hat{z} - \sin \xi \hat{y} \). The corresponding unit vector from the star towards the observer is \( \hat{e} = -\sin i \hat{i} + \cos i \hat{z} \). Our approach is to express everything in the inertial coordinate system, then find the sub-stellar point with appropriate dot products.

For the planetary surface, we use a second coordinate system fixed to the planet. We align the \( z_p \)-axis with the rotational angular frequency, \( \hat{z}_p = \hat{\omega}_{\text{rot}} \), while the \( x_p \)-axis is set by our choice for the prime meridian (and initial sub-observer longitude.) The final axis, \( y_p \), is again determined by taking \( \hat{y}_p = \hat{z}_p \times \hat{x}_p \). We proceed in two steps, first finding the planetary axes at \( t = 0 \), then using the planet’s rotation to describe these axes at any time.

Since we disregard precession, the planet’s rotation axis is time-independent:

\[
\hat{z}_p = \hat{\omega}_{\text{rot}} = -\cos \xi_s \sin \Theta \hat{z} - \sin \xi_s \sin \Theta \hat{y} + \cos \Theta \hat{\xi}. \tag{A7}
\]

The sub-observer point is on the prime meridian when \( t = 0 \), so that

\[
\hat{y}_p(0) = \hat{y}_p = \hat{z}_p \times \hat{x}_p. \tag{A8}
\]

Computing this we find

\[
\hat{y}_p(0) = \frac{1}{\sin \theta_s} \left[ -\cos i \xi_s \sin \Theta \hat{\xi} \right. \\
+ (\cos i \xi_s \sin \Theta - \sin i \cos \Theta) \hat{y} \\
\left. - i \sin \xi_s \sin \Theta \hat{\xi} \right]. \tag{A9}
\]

The starting \( x_p \)-axis is then found by taking \( \hat{x}_p(0) \times \hat{z}_p \). The result is sim-
This results in the relations
\[ \sin \Theta \sin \phi \cos \theta_0 - \sin \phi \cos i \sin \phi \theta_0 = \sin i \sin \phi \cos \theta_0, \]
\[ + \sin \phi (\cos \sin \theta_0 - \sin i \cos \phi \cos \theta_0) \xi. \]

We can now find the planetary axes, in terms of the inertial axes, at any time by rotating Equations A9 and A10 about the \( z_p \)-axis:
\[ \hat{x}_p = \cos(\omega t) \hat{x}_p(0) + \sin(\omega t) \hat{y}_p(0), \]
\[ \hat{y}_p = -\sin(\omega t) \hat{x}_p(0) + \cos(\omega t) \hat{y}_p(0). \]
The sub-stellar angles in the planetary coordinates may then be extracted from the relations
\[ \sin \theta_s \cos \phi_s = \hat{r}_p \cdot \hat{x}_p, \]
\[ \sin \theta_s \sin \phi_s = \hat{r}_p \cdot \hat{y}_p, \]
\[ \cos \phi_s = \hat{r}_p \cdot \hat{z}_p, \]
resulting in:
\[ \cos \theta_s = \sin \Theta \cos [\xi - \xi], \]
\[ \sin \theta_s = \sqrt{1 - \sin^2 \Theta \cos^2 [\xi - \xi]}, \]
\[ \cos \phi_s = \cos(\omega t) \cos \theta_0 + \sin(\omega t) \sin \phi, \]
\[ \sin \phi_s = -\sin(\omega t) \cos \theta_0 + \cos(\omega t) \sin \phi, \]
where the factors \( a(t) \) and \( b(t) \) are given by
\[ a(t) = \{ \sin i \cos \xi - \cos \theta_0 \sin \Theta \cos [\xi - \xi] \}, \]
\[ b(t) = \{ \sin i \sin \xi \cos \Theta - \sin \phi \sin \Theta \sin [\xi - \xi] \}. \]
Note that when \( \theta_0 = 0, \pi \), the sub-stellar longitude can be set arbitrarily to avoid dividing by zero in Equations A18 and A19.

### A2 Polar Observer

Equations A18 and A19 for the sub-stellar longitude apply to most observers. However, the definition of \( \hat{y}_s(0) \) in Equation A8 fails when the sub-observer point coincides with one of the planet’s poles. Two alternate definitions can be used in these situations.

**Case 1:** If the sub-stellar point will not pass over the poles during orbit, we may define
\[ \hat{y}_s(0) = -\hat{y}, \]
so that
\[ \hat{x}_s(0) = \hat{y}_s(0) \times \hat{x}_p = -\cos \Theta \hat{x} - \cos \xi \sin \Theta \hat{z}. \]
This results in
\[ \cos \phi_s = \frac{\cos \omega t \cos \xi \cos \Theta + \sin \omega t \sin \xi}{\sqrt{1 - \sin^2 \Theta \cos^2 [\xi - \xi]}}, \]
\[ \sin \phi_s = \frac{-\sin \omega t \cos \xi \cos \Theta + \cos \omega t \sin \xi}{\sqrt{1 - \sin^2 \Theta \cos^2 [\xi - \xi]}}. \]

**Case 2:** However, if the sub-stellar point will pass over the poles during orbit, we define instead
\[ \hat{x}_s(0) = \hat{z}, \]
such that
\[ \hat{y}_s(0) = \hat{z}_p \times \hat{x}_p = \cos \xi \hat{y}. \]
This produces
\[ \cos \phi_s = \frac{-\sin \omega t \sin \xi \cos \xi}{\sqrt{1 - \sin^2 \Theta \cos^2 [\xi - \xi]}}. \]

These special cases only impact the sub-stellar longitude: expressions for the other angles are unchanged. As with a general observer, the Case 2 sub-stellar longitude may be set arbitrarily whenever \( \theta_0 = 0, \pi \).

### A3 Zero Obliquity

For non-oblique planets, \( \Theta = 0^\circ \), the sub-observer colatitude satisfies
\[ \cos \theta_0 = \cos i, \]
\[ \sin \theta_0 = \sin i, \]
while the sub-stellar angles become
\[ \cos \theta_s = 0, \]
\[ \sin \theta_s = 1, \]
\[ \cos \phi_s = \frac{\cos(\omega t) \cos \xi + \sin(\omega t) \sin \xi}{\sqrt{1 - \cos^2 i}}, \]
\[ \sin \phi_s = -\sin(\omega t) \cos \xi + \cos(\omega t) \sin \xi, \]
where \( c(t) \) and \( d(t) \) are given by
\[ c(t) = \sin i \cos \phi, \]
\[ d(t) = \sin i \sin \phi. \]

The sub-stellar longitude is therefore
\[ \cos \phi_s = \frac{\cos(\omega t) \sin i \cos \xi + \sin(\omega t) \sin i \sin \xi}{\sqrt{1 - \cos^2 i}}, \]
\[ = \cos(\omega t) \cos \xi + \sin(\omega t) \sin \xi \]
\[ = \cos(\xi - \omega t), \]
\[ \sin \phi_s = -\sin(\omega t) \cos \xi + \cos(\omega t) \sin \xi \]
\[ = -\sin(\xi - \omega t), \]
In other words, \( \theta_0 = i, \phi_0 = \phi_0(0) - \omega t, \theta_0 = 0, \) and \( \phi_s = \xi - \omega t \).

### APPENDIX: KERNEL DETAILS

#### B1 Characteristics

An important measure of the longitudinal kernel is its width, \( \sigma_\phi \), as shown in the left panel of Figure B1. We treat this width mathematically as a standard deviation. Since \( K(\phi, 0) \) is on a periodic domain, we minimize the variance for each geometry with respect to the grid location of the prime meridian, \( \phi_0 \):
\[ \sigma_\phi^2 = \min_0^{2\pi} \left[ \int_0^{2\pi} (\phi - \phi_0)^2 K(\phi) d\phi \right] \phi_0, \]
where \( K(\phi) \) is the spherically normalized longitudinal kernel, \( \phi \equiv \phi + \phi_0, \) and \( \phi \) is the mean longitude:
\[ \phi = \int_0^{2\pi} \phi K(\phi) d\phi. \]

All longitude arguments and separations in Equations B1 and B2 wrap around the standard domain \([0, 2\pi]\). Also note the unprimed arguments inside the kernel: these make computing the variance simpler. The minimum variance determines the standard deviation of the kernel, and thus width, for a given geometry.
The dominant colatitude is similarly important for the latitudinal kernel, as shown in the right panel of Figure B1. Cowan et al. (2012) defined the dominant colatitude, \( \bar{\theta} \), for a given geometry:

\[
\bar{\theta} = \int_0^{\pi} \theta K(\theta, \phi) \sin \theta d\theta,
\]

Equation B3 is equivalent to

\[
\bar{\theta} = \int_0^{\pi} \theta \hat{K}(\theta) \sin \theta d\theta,
\]

where \( \hat{K}(\theta) = K(\theta) / f K(\theta) \sin \theta d\theta \) is the spherically normalized latitudinal kernel. The dominant colatitude is the North-South region that gets sampled most by the kernel (e.g. the circles in Figure B1.)

### B2 Albedo Variations

Figure 1 demonstrates that obliquity is related to—though not explicitly given by—both rotational and orbital variations of a planet’s apparent albedo. We use a Monte Carlo approach to quantify these relations, simulating planets with different maps and viewing geometries. We generate albedo maps from spherical harmonics, \( Y_{\ell m}(\theta, \phi) \), on the same 101 \times 201 grid in colatitude and longitude from Section 2.2:

\[
A(\theta, \phi) = \sum_{\ell=0}^{\ell_{\text{max}}} \sum_{m=-\ell}^{\ell} C_{\ell m}^\theta Y_{\ell m}(\theta, \phi),
\]

where \( \ell_{\text{max}} \) is chosen to be 3, each coefficient \( C_{\ell m}^\theta \) is randomly drawn from the standard normal distribution, and the composite map is scaled to the Earth-like range [0.1, 0.8]. Rotational and orbital changes in brightness are caused by East-West and North-South albedo markings, respectively, so we make three types of maps: East-West featured with \( C_{\ell m}^\theta (m \neq \ell) = 0 \), North-South featured with \( C_{\ell m}^\theta (m \neq 0) = 0 \), or no \( C_{\ell m}^\theta \) restrictions. For all maps with East-West features, we randomly offset the prime meridian. We generate 5,000 maps of each type.

For each map we randomly select an obliquity, solstice phase, inclination, and two orbital phases. Since inclination and orbital phase can be measured independent of photometry, we choose inclinations similar to planet \( \Psi, i \in [50^\circ, 70^\circ] \), and orbital phases \( \{\xi_1, \xi_2\} \) with \( \Delta \xi \in [110^\circ, 130^\circ] \). Both phases are also at least \( 30^\circ \) from superior and inferior conjunction, which conservatively mimics an inner working angle at the selected inclinations. We assume the planet’s rotational and orbital frequencies are known (Pullé et al. 2008; Oakley & Cash 2009), and use the Earth-like ratio \( \omega_{\text{rot}} / \omega_{\text{orb}} = 360 \). We divide roughly one planet rotation centered on each orbital phase into 51 time steps, then define the normalized amplitude of rotational and orbital albedo variations, \( \Lambda_{\text{rot}} \) and \( \Lambda_{\text{orb}} \), as:

\[
\Lambda_{\text{rot}} = A_{\xi_1}^{\text{high}} - A_{\xi_1}^{\text{low}},
\]

\[
\Lambda_{\text{orb}} = |\bar{A}_{\xi_1} - A_{\xi_1}| \left( \frac{\bar{A}_{\xi_1} + A_{\xi_1}}{2} \right)^{-1},
\]

where \( A_{\xi_1}^{\text{high}} \) and \( A_{\xi_1}^{\text{low}} \) are the extreme apparent albedos around the first phase, and \( \bar{A} \) is the mean apparent albedo of all time steps around a given phase. For each computed \( \Lambda_{\text{rot}} \) and \( \Lambda_{\text{orb}} \), we calculate the corresponding kernel width and absolute value change in dominant colatitude, from Appendix B1. Figure B2 shows the resulting distributions, where rotational and orbital information is colored red and blue, respectively. We find similar results when relaxing constraints on the inclination and orbital phases.

We can estimate uncertainties on values of \( \sigma_\theta \) and \( \Delta \theta \) using these distributions. The mean rotational and orbital variations are approximately 0.54 and 0.21 each; the average kernel width and change in dominant colatitude are both roughly 38°. For the full distributions we find standard deviations of about 17° on \( \sigma_\theta \) and 24° on \( \Delta \theta \), but roughly 5° and 7° respectively for large variations. To demonstrate constraints on obliquity, we consider a typical scenario and assume we have single-epoch observations, explicitly at \( \xi(t) = (120^\circ, 300^\circ) \), with the mean variations (0.54 and 0.21) from our distributions. The sets of samples near these variations show about 10° and 20° standard deviations apiece in the kernel width and change in dominant colatitude, which we use as example uncertainties in Figure 6.

### B3 Peak Motion

Equations C1 and C2 from Cowan et al. (2009) describe the motion of the kernel peak, where specular reflection occurs, for any planetary system. These equations can be written for edge-on, zero obliquity planets using Section A3:

\[
\cos \theta_{\text{spec}} = \frac{1 + \cos i}{\sqrt{2(1 + \cos i)}}.
\]
When finding $\phi_{\text{spec}}$ from Equation B9, the two-argument arctangent must be used to ensure $\phi_{\text{spec}} \in [−\pi, \pi)$. This also means it is difficult to simplify the equation with trigonometric identities.

Instead, we can explicitly write the argument of Equation B9 in terms of the first meridian crossing, $\xi_m$, the earliest orbital phase after superior conjunction that the kernel peak recrosses the prime meridian:

$$\phi_{\text{spec}}(\xi; \xi_m) = \pm \frac{\pi}{2} \left( \frac{4}{\xi_m} \xi + \left[ 1 - \text{sgn} \left( \cos \frac{\xi}{2} \right) \right] \right),$$

(B10)

where the leading upper sign applies to prograde rotation and vice versa. The first meridian crossing is related to the planet’s frequency (or period) ratio by

$$\left| \frac{\omega_{\text{rot}}}{\omega_{\text{orb}}} \right| = \frac{P_{\text{rot}}}{P_{\text{orb}}} = \frac{1}{2} \left( \frac{4\pi}{\xi_m} \pm 1 \right),$$

(B11)

while the number of solar days per orbit is

$$N_{\text{solar}} = \left| \frac{\omega_{\text{rot}}}{\omega_{\text{orb}}} \right| \mp 1,$$

(B12)

following the same sign convention. Note that the frequency/period ratios and the number of solar days do not have to be integers. We reiterate that Equations B8 through B12 apply to edge-on, zero obliquity planets.

Equation B11 gives two frequency ratios for each first meridian crossing, one prograde and another retrograde that is smaller in magnitude by unity. Equation B12 then says the corresponding difference in solar days is unity but reversed, making the longitudes of both kernel peaks in Equation B10 analogous at each orbital phase. These two versions of the planet have East-West mirrored albedo maps and identical light curves: they are formally degenerate. An inclined, oblique planet has pro/retrograde versions that could be distinguished, as discussed in Section 4.4.

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