

Now oriented simplexes. Look at e.g. 1-simplex.

$$\langle p_0 p_1 \rangle = \begin{array}{c} p_1 \\ \diagdown \\ p_0 \end{array} = \langle p_1 p_0 \rangle \text{ unordered.}$$

Change of notation, write $(p_0 p_1)$ for ordered simplex,

$$(p_0 p_1) = \begin{array}{c} p_1 \\ \nearrow \\ p_0 \end{array} = -(p_1 p_0).$$

For 2-simplexes, have 3 points, $(p_0 p_1 p_2)$. Declare that this changes sign if points subjected to an odd permutation. Generally,

$$(p_{i_0} p_{i_1} \dots p_{i_n}) = \pm (p_0, \dots, p_n)$$

where \pm = sign of permutation $\begin{pmatrix} 0 & 1 & \dots & n \\ i_0 & i_1 & \dots & i_n \end{pmatrix}$.

Special case of 0-dimension. $\langle p_0 \rangle = \bullet$ a point.

(p_0) = "oriented" point. ~~so~~

As discussed previously, motivation for oriented simplexes is use in integrals, e.g., Stokes' thm. Here is what we mean by a "0-dimensional integral":

$$\int_{(p_0)} f = f(p_0) \quad \text{where } p_0 \in M$$

$f: M \rightarrow \mathbb{R}$ (say).

Thus,

$$\int_{-(p_0)} f = -f(p_0), \text{ etc.}$$

- Thus linear combinations of oriented simplexes become meaningful (as things you integrate over).

Above we defined a simplicial complex K as a collection of unoriented simplexes. Now we modify the definition in obvious ways to talk about an oriented simplicial complex K : it is a set of oriented simplexes $\{\sigma_i\}$. (Use the same symbols.)

Let $K = \text{an oriented simplicial complex} = \{\sigma_i\}$. Define:

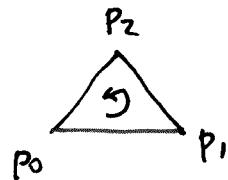
Def: The r -th chain group $C_r(K)$ is the free Abelian group generated by the r -dimensional, oriented simplexes in K , that is, it is the set of formal linear combinations,

$$c = \sum_i^{I_r} n_i \sigma_{ri}, \quad n_i \in \mathbb{Z}, \quad c \in C_r(K).$$

where $\{\sigma_{ri}\}$ is the set of r -simplexes in K , whose number is I_r .

c is called an r -chain. Note, $C_r(K) \cong \mathbb{Z}^{I_r}$.

Now we motivate the definition of the boundary operator. Start with an oriented 2-simplex (p_0, p_1, p_2)



The arrow is a convenient way of specifying this orientation, since any cyclic permutation of (p_0, p_1, p_2) is an even permutation. (The opposite direction would reverse the sign.) We define the boundary operator in this example by writing

$$\partial \begin{array}{c} p_2 \\ \swarrow \searrow \\ p_0 & p_1 \end{array} = \begin{array}{c} p_2 \\ \nearrow \searrow \\ p_0 & p_1 \end{array} \quad \text{or}$$

$$\partial (p_0, p_1, p_2) = (p_0 p_1) + (p_1 p_2) + (p_2 p_0).$$

Notice that ∂ acting on a 2-simplex is not a simplex, but rather a linear combination of simplexes (a chain). Another example,
 • the boundary of a 1-simplex:

$$\partial(p_0 p_1) = (p_1) - (p_0)$$

$$\partial \begin{array}{c} p_1 \\ \diagdown \\ p_0 \end{array} = \begin{array}{c} p_1 \\ \cdot \\ -p_0 \end{array}$$

In general, we define $\partial_r: C_r(K) \rightarrow C_{r-1}(K)$ by giving its action on an r -simplex and then extending in the obvious way to linear combinations. If $\sigma_r = (p_0, \dots, p_r)$ is an oriented r -simplex, then we define

$$\partial_r \sigma_r = \sum_{i=0}^r (-1)^i (p_0 \dots \hat{p}_i \dots p_r)$$

↙ hat means omit this point.

E.g., $\partial_2(p_0 p_1 p_2) = (p_1 p_2) - (p_0 p_2) + (p_0 p_1)$, same answer as before since $-(p_0 p_2) = +(p_2 p_0)$.

One more note about defn of bdry operator. In special case $r=0$, we define

$$\partial_0(p_0) = 0,$$

since there are no (-1) -chains.

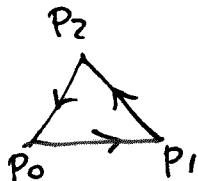
Also note, $\partial_r: C_r(K) \rightarrow C_{r-1}(K)$ is a group homomorphism (it commutes with $+$).

There is really one boundary operator for each dimension. Each maps $C_r(K)$ into $C_{r-1}(K)$. So we have a sequence of maps, ($n = \dim K$)

$$C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} \{0\}.$$

Def: An r -chain $c \in C_r(K)$ such that $\partial c = 0$ is called a cycle (or r -cycle). A cycle is a chain without a boundary.

Example:



$$c = (p_0 p_1) + (p_1 p_2) + (p_2 p_0)$$

$$\partial c = (p_1) - (p_0) + (p_2) - (p_1) + (p_0) - (p_2) = 0.$$

c is a cycle (1 -cycle).

- Another example: Any 0 -chain is a cycle, since $\partial(p_0) = 0$.

Def: The set

$$Z_r(K) = \{c \in C_r(K) \mid \partial_r c = 0\} = \ker \partial_r$$

is the r -th cycle group. It is obviously a subgroup of $C_r(K)$, which means (see notes with HW³) that it is a sublattice of $C_r(K)$ and that it (like $C_r(K)$) is a free, finitely generated Abelian group.

Special case: $Z_0(K) = C_0(K)$ (all 0 -chains are cycles).
($r=0$)

Def: The set

$$B_r(K) = \{ b \in C_r(K) \mid b = \partial_{r+1} c, \text{ some } c \in C_{r+1}(K) \} = \text{im } \partial_{r+1}$$

is the r -th boundary group. Elements of $B_r(K)$ are called r -boundaries. $B_r(K)$ is obviously a subgroup of $C_r(K)$, hence it is a free, finitely generated Abelian group.

Special case: For $r=n$, since there are no $(n+1)$ -simplices, we define $B_n(K) = \text{"im } \partial_{n+1}" = \{0\}$.

Thm: $\partial_r \partial_{r+1} = 0$. The boundary of a boundary vanishes.

Proof: It suffices to consider an $(n+1)$ -simplex (a basis vector in $C_{n+1}(K)$), (p_0, \dots, p_{n+1}) :

$$\partial_{n+1}(p_0, \dots, p_{n+1}) = \sum_{i=0}^{n+1} (-1)^i (p_0 \dots \hat{p}_i \dots p_{n+1})$$

$$\begin{aligned} \partial_n \partial_{n+1}(p_0, \dots, p_{n+1}) &= \sum_{i=0}^{n+1} (-1)^i \left[\sum_{j=0}^{i-1} (-1)^j (p_0 \dots \hat{p}_j \dots \hat{p}_i \dots p_{n+1}) \right. \\ &\quad \left. + \sum_{j=i+1}^{n+1} (-1)^{j+1} (p_0 \dots \hat{p}_i \dots \hat{p}_j \dots p_{n+1}) \right] = 2 \text{ terms} \end{aligned}$$

~~Swap i, j in 2nd term, it becomes~~

$$-\sum_{i=0}^{n+1} (-1)^i \sum_{j=}$$

$$\text{1st term} = \sum_{\substack{i+j \\ i > j}} (-1)^{i+j} (p_0 \dots \hat{p}_j \dots \hat{p}_i \dots p_{r+1})$$

$$\text{2nd term} = - \sum_{\substack{i+j \\ i < j}} (-1)^{i+j} (p_0 \dots \hat{p}_i \dots \hat{p}_j \dots p_{r+1}) = - \text{1st term by swapping } i, j.$$

QED.

immediate corollary: $B_r(K) \subseteq Z_r(K)$.

if $b \in B_r(K)$, then $b = \partial c$, some $c \in C_{r+1}(K)$.

Hence $\partial b = \partial \partial c = 0$ hence $b \in Z_r(K)$.

Altogether,

$$B_r(K) \subseteq Z_r(K) \subseteq C_r(K)$$

Actually \subseteq means "subgroup". all 3 groups are free, finitely generated Abelian group.

Finally, we define

\leftarrow Note, for $r > \dim K$, $H_r(K)$ is understood to be $\{0\}$ (the trivial group).

$$H_r(K) = \frac{Z_r(K)}{B_r(K)} = r\text{-th homology group.}$$

$H_r(K)$ is a topological invariant, that is, if $|K|$ and $|K'|$ are homeomorphic, then $H_r(K) \cong H_r(K')$. If you have a topological space X homeomorphic to $|K|$, then $H_r(X)$ is regarded as the same as $H_r(K)$.

Note, if $h \in H_r(K)$, then h is an equivalence class of cycles whose difference is a boundary,

$$h = [c], \quad c \in Z_r(K),$$

$$c \sim c' \text{ if } c - c' \in B_r(K).$$

Note that the zero element in $H_r(K)$ is the equivalence class of boundaries,

$0 \in H_r(K)$ means

$$[0] = Br(K).$$

Some general features of homology groups. First take case $r=0$.

As noted above, all 0-simplexes (p) are automatically cycles.

Note that the boundary of a 1-simplex is always the difference between the endpoints,

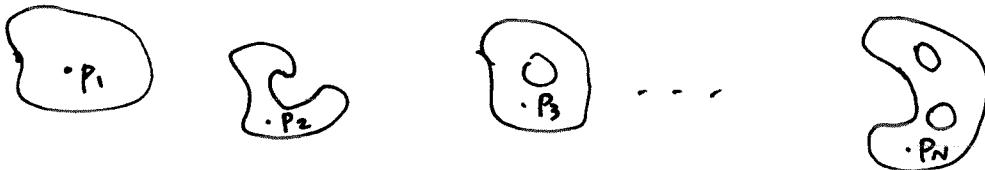
$$\partial \left(\begin{smallmatrix} p & q \\ \curvearrowright & \curvearrowright \end{smallmatrix} \right) = (q) - (p),$$

(p) and (q)

which shows that any two 0-simplexes are homologous if the points p and q can be connected by a curve.

$$(p) = (q) \Leftrightarrow [(q) - (p)].$$

In fact this is iff. So if a manifold M consists of N disconnected pieces,



Then all 0-simplexes (p) where p belongs to one piece are homologous to all other simplexes (q) where q belongs to the same piece. So the equivalence classes of cycles are generated by $[(p_1)], \dots, [(p_N)]$ where p_1, \dots, p_N are taken from each piece. Thus, a general element of $H_0(M)$ has the form,

$$h \in H_0(M),$$

$$h = \sum_{i=1}^N n_i [(p_i)],$$

and $H_0(M) = \mathbb{Z}^N$, $N = \# \text{ of disconnected pieces of } M$.

In particular, for a connected manifold, $H_0(M) = \mathbb{Z}$.

Review, $B_r(K) \subseteq Z_r(K) \subseteq C_r(K)$

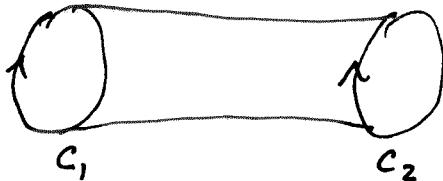
$$H_r(K) = \frac{Z_r(K)}{B_r(K)}. \quad (\text{cycles modulo boundaries})$$

Special case $r=0$: $Z_0(K) = C_0(K)$, hence $H_0(K) = \frac{C_0(K)}{B_0(K)}$.

In fact, $H_0(K) = \mathbb{Z}^N$, $N = \# \text{ of connected components of } K$.

Special case $r=n=\dim K$. $B_n(K) = \{0\}$, $H_n(K) = Z_n(K)$.

A general principle (connection between homology and homotopy). Suppose we have two cycles that can be continuously deformed into one another ("freely homotopic"). Picture for example two 1-cycles,



As C_1 deforms into C_2 it sweeps out a 2-chain d , with $C_1 - C_2 = \partial d$. Hence C_1 and C_2 are homologous. Of course, in our approach to homology theory, the 1-cycles and the 2-chain should be made up out of the simplexes of a triangulation. But the principle is the same:

Two cycles that can be continuously deformed into one another (are freely homotopic) are homologous.

The converse however is not true, see the HW.

Now do some examples of homology groups.

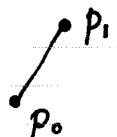
1. single point, $\bullet p_0$, $K = \{(p_0)\}$.

$$H_0(K) = \mathbb{Z} \quad (\text{one connected component}).$$

$$H_r(K) = \{0\}, \quad r \geq 1.$$

We always understand $H_r(K) = \{0\}$ for $r > n = \dim K$.

2. a line segment.



$$K = \{(p_0 p_1), (p_0), (p_1)\}.$$

$$H_0(K) = \mathbb{Z} \quad (\text{one connected component}).$$

$$H_1(K) = \frac{\mathbb{Z}_1(K)}{B_1(K)} = \mathbb{Z}_1(K) \quad \text{since } B_1(K) = \{0\}.$$

so we need $\mathbb{Z}_1(K)$. First what is $C_1(K)$? There is only one 1-simplex in K , $(p_0 p_1)$, so

$$C_1(K) = \text{gen}\{(p_0 p_1)\} = \{n(p_0 p_1) \mid n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

All 1-chains are just integer multiples of $(p_0 p_1)$. See which ones of these are ~~homotopies~~ cycles.

$$\partial[n(p_0 p_1)] = n \partial(p_0 p_1) = n[(p_1) - (p_0)] = 0 \quad (\text{demand}).$$

But $(p_0), (p_1)$ are independent, so this $\Rightarrow n=0$. The only 1-cycle is 0, hence $\mathbb{Z}_1(K) = \{0\}$, and

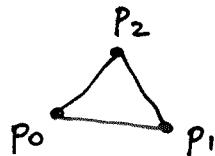
$$H_1(K) = \{0\}.$$

Beware, Nakahara writes \mathcal{O} instead of $\{0\}$. Thus, for example, he would write

$$H_1(K) = \frac{\{0\}}{\{0\}} = \frac{\mathbb{Z}_1(K)}{\mathbb{B}_1(K)} \text{ in this example}$$

He would write $\frac{\mathcal{O}}{\mathcal{O}}$ instead of $\frac{\{0\}}{\{0\}}$. The latter (correct) expression is perfectly well defined; each group contains one element.

3. A triangle,
 ↑
 (the boundary of)



$$n = \dim K = 1.$$

$$K = \{(p_0 p_1), (p_1 p_2), (p_2 p_0), (p_0), (p_1), (p_2)\}.$$

Note, $|K|$ is homeomorphic to a circle S^1 .

Again, case $r=0$ is easy, $H_0(K) = \mathbb{Z}$.

For $r=1$ we have $H_1(K) = \mathbb{Z}_1(K)$. Intuitively obvious that there is one 1-cycle, $(p_0 p_1) + (p_1 p_2) + (p_2 p_0)$, , and that all 1-cycles are linear combinations of ~~this~~ this one (i.e. multiples of this one). But to prove it, first look at $C_1(K)$:

$$C_1(K) = \text{gen}\{(p_0 p_1), (p_1 p_2), (p_2 p_0)\}$$

$$= \left\{ \cancel{n_2(p_0 p_1) + n_0(p_1 p_2) + n_1(p_2 p_0)} \mid n_0, n_1, n_2 \in \mathbb{Z} \right\} \cong \mathbb{Z}^3.$$

Now demand that an arbitrary element of $C_1(K)$ be a cycle:

$$\begin{aligned} 0 &= \partial [n_2(p_0 p_1) + n_0(p_1 p_2) + n_1(p_2 p_0)] \\ &= (n_1 - n_2)(p_0) + (n_2 - n_0)(p_1) + (n_0 - n_1)(p_2) \end{aligned}$$

$$= 0 \quad \text{iff} \quad \left. \begin{array}{l} n_1 = n_2 \\ n_2 = n_0 \\ n_0 = n_1 \end{array} \right\} \quad \text{i.e.} \quad n_0 = n_1 = n_2.$$

(11)

So,

$$Z_1(K) = \text{gen}\{(P_0P_1) + (P_1P_2) + (P_2P_0)\} \cong \mathbb{Z},$$

as expected, and

$$H_1(K) = \mathbb{Z}.$$

Note the difference between S' and Δ : both are 1-dimensional, but

$$H_1(S') = \{0\}$$

$$H_1(\Delta) = \mathbb{Z}.$$

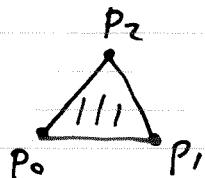
The \mathbb{Z} in the second case reflects the existence of one cycle that is not a boundary. It indicates the existence of a "hole" in the space.

Note also that Δ is homeo. to S' , so we now have the homology groups for S' :

$$H_0(S'_*) = \mathbb{Z},$$

$$H_1(S') = \mathbb{Z}.$$

4. A triangle (the full 2D object):



$$K = \{(P_0P_1P_2), (P_0P_1), (P_1P_2), (P_2P_0), (P_0), (P_1), (P_2)\}.$$

Same K as in last example, except $(P_0P_1P_2)$ now included.

So, $\dim K = n = 2$.

Again $H_0(K) = \mathbb{Z}$. ($r=0$). Next, $r=2$.

$H_2(K) = \mathbb{Z}_2(K)$. The only 2-chain is a multiple of $(P_0 P_1 P_2)$, but $\partial(P_0 P_1 P_2) = (P_0 P_1) + (P_1 P_2) + (P_2 P_0) \neq 0$, so the only 2-cycle is 0 and $H_2(K) = \{0\}$.

Now $r=1$.

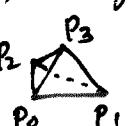
$$H_1(K) = \frac{\mathbb{Z}_1(K)}{B_1(K)}.$$

First look at $\mathbb{Z}_1(K)$. It is the same as in the previous example ($C_1(K)$ is the same, too), since $C_1(K)$ is generated by same generators. But $B_1(K)$ is different, since now there are 2-chains whose bdry we can take. In fact, from above we have

$$B_1(K) = \text{gen}\{(P_0 P_1) + (P_1 P_2) + (P_2 P_0)\} = \mathbb{Z}_1(K) \cong \mathbb{Z}.$$

$$\text{So, } H_1(K) = \frac{\mathbb{Z}_1(K)}{B_1(K)} \cong \frac{\mathbb{Z}}{\mathbb{Z}} \cong \{0\}.$$

The reason $H_1(K)$ changed from \mathbb{Z} to $\{0\}$ on going from example [3] to [4] is because the hole disappeared.

- [5]. A tetrahedron (i.e., surface of a tetrahedron), a 2-dim simplicial complex, Δ 

We'll use some intuition and general principles.

First, $r=0$. Easy, $H_0(K) = \mathbb{Z}$.

Next, $r=2$. $H_2(K) = \mathbb{Z}_2(K)$. Are there any 2 cycles?

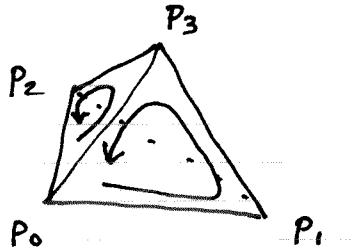
Intuitively, a 2-cycle is a "closed surface", and the sphere S^2 is certainly one of these (hence also the tetrahedron).

Thus we expect that $\partial(S^2) = 0$, where (S^2) stands for the 2-chain that is the whole surface of the tetrahedron, actually the sum of 4 2-simplexes,

$$(S^2) = (P_0 P_3 P_2) + (P_0 P_1 P_3) + (P_2 P_3 P_1) + (P_0 P_2 P_1).$$

Here each of the faces has been oriented by the "right hand rule" for an "outward pointing normal". You can check directly that $\partial(S^2) = 0$.

Another way is to note that the common edges of any 2 adjacent triangles is oppositely oriented by those triangles,



so the common edge cancels when we take the boundary. In fact, the most general 2-chain is a linear combination of these 4 triangles, and the only way that its boundary can vanish is if the coefficients of all adjacent triangles (hence all triangles) are equal. Thus, every 2-cycle is a multiple of (S^2) itself, confirming our intuition. Hence $Z_2(K) = \mathbb{Z} = H_2(K)$.

Finally, case $r=1$. Question: Are there any 1-cycles that are not boundaries? If you liberate yourself from the triangulation, the answer is no (you can see), because any closed loop on S^2 is clearly a boundary:



Alternatively, note that all cycles on the sphere S^2 can be contracted to a point (the zero 1-chain), so therefore are boundaries. Hence $\mathbb{Z}_1(K) = B_1(K)$ and $H_1(K) = \{0\}$.

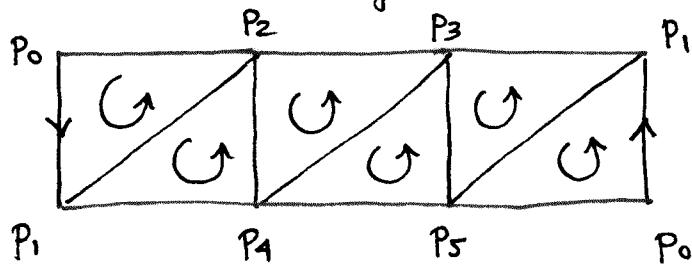
Notice two examples of spheres so far, dim 1 and 2:

	$H_0(K)$	$H_1(K)$	$H_2(K)$
S^1	\mathbb{Z}	\mathbb{Z}	
S^2	\mathbb{Z}	$\{0\}$	\mathbb{Z} .

\swarrow means S^2 is "closed".

Can generalize, show that for S^n , $H_0(S^n) = \mathbb{Z} = H_n(S^n)$, and $H_r(S^n) = \{0\}$ for $0 < r < n$. Intuitively the reason for the ~~vanishing~~^{triviality} of $H_r(S^n)$ for $1 \leq r \leq n-1$ is the same as that for $H_1(S^2)$: The r -cycles on the face of S^n are contractible to a point.

- 6 The Möbius strip. Regard it as a square with opposite sides identified (in reverse order), and triangulate it as we did the cylinder.

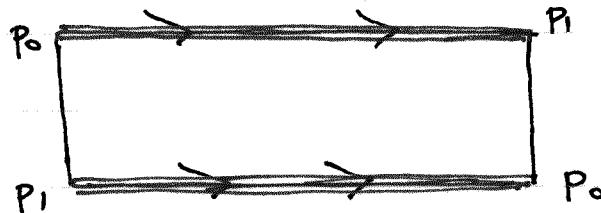


orient the triangles as shown.

Case $r=0$ easy (as always), $H_0(K) = \mathbb{Z}$.

Now case $r=2$. For sphere we found $\partial(S^2) = 0$. What is $\partial(Mö)$, where $(Mö)$ is the whole Möbius strip, the sum of all 6 ~~rectangles~~ triangles (2-simplexes) as shown? Unlike the sphere, $Mö$ has an edge (only one, actually), so that if a triangle has an edge that is part of the edge of $Mö$, then that edge (which appears when you apply ∂ to $(Mö)$) can never be cancelled by an adjacent triangle. (The same would be true for any space with a boundary, e.g., the cylinder). In fact, all triangles in $(Mö)$ have at least one edge that is a part of the edge of $(Mö)$ (there are no purely "internal" triangles).

A guess for what the edge of $Mö$ would be is



Heavy lines are the path.

i.e., $p_0 \rightarrow p_1 \rightarrow p_0$ again along the obvious "edge" you would see if you constructed a Möbius strip out of paper. But this is not what you get when you take $\partial(Mö)$ where

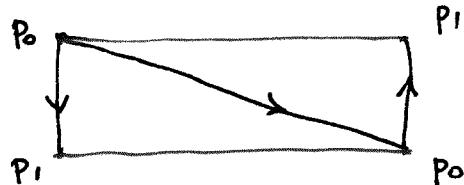
$$(Mö) = \sum_{\text{shown}} (\text{six 2-cycles}),$$

i.e. the six triangles with the orientation shown. Instead you get

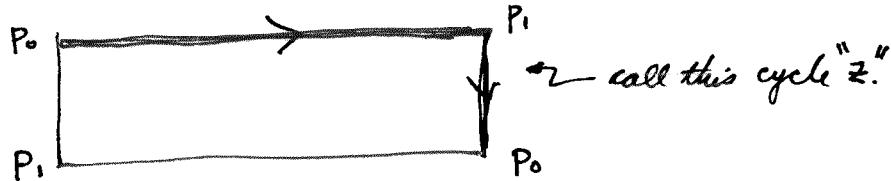
$$\partial(Mö) = \begin{array}{c} \text{Diagram of a rectangle with arrows on all four edges showing orientation.} \end{array} = \begin{array}{c} \text{Diagram of a rectangle with arrows on the top and bottom edges pointing right, and the left and right sides pointing left.} \end{array} + 2 \left(\begin{array}{c} \text{Diagram of a triangle with arrows on all three edges pointing clockwise.} \\ p_1 \\ p_0 \end{array} \right).$$

So there are 2 reasons why $\partial(\text{Mö})$ does not vanish: Mö has an edge, so abutting triangles have edges that can't be cancelled, and secondly because even the internal edges cannot be all cancelled because you can't orient the triangles "coherently" (so that all internal edges cancel due to opposite orientations of adjacent triangles). The latter effect is due to the non-orientability of the Möbius strip. In fact, it's easy to see that no linear combination of any set of triangles with any orientation can give a 2-cycle. There are no 2-cycles, $\mathbb{Z}_2(K) = \{0\}$ and $H_2(K) = \{0\}$.

Now case $r=1$. $H_1(K)$ of course contains the element 0, the equivalence class of 1-cycles that are also boundaries, for example the boundary of any triangle. Are there one-cycles that are not boundaries? Inspection suggests one candidate,



(the diagonal line). This is drawn without regard to the triangulation, but by contin. deformation we can make it run along the available 1-simplices (edges):



This cycle is not a boundary. If it were, it would have to be the boundary of a sum of triangles that at a minimum would include those with edges along the top:

