

Physics 250
Fall 2015
Homework 8
Due Friday, October 23, 2015

Reading Assignment: Nakahara, pp. 196–206. See also Frankel, pp. 52–84, 125–136, 145–154.

Notes. In Eq. (5.62), p. 196 (2nd ed), the wedge products on the right hand side should be replaced by tensor products. Equation (5.83), p. 202, is meaningless. Please ignore it. Also, Eq. (5.89), p. 203, should read $\theta = p_\mu dq^\mu$. Otherwise the material on pp. 191–204 is ok.

1. (DTB) Nakahara Exercise 5.15, p. 199 (Exercise 5.32, p. 161 of the first edition).

2. (DTB) As was discussed in class, the Lie derivative L_X obeys the Leibnitz rule when acting on tensor products. As was also discussed, the exterior product \wedge is an antisymmetrized tensor product.

(a) Let $\alpha \in \Omega^r(M)$ and $\beta \in \Omega^s(M)$. Find an expression for $L_X(\alpha \wedge \beta)$ in terms of $L_X\alpha$ and $L_X\beta$.

(b) The *Cartan formula* is

$$L_X = i_X d + di_X, \quad (8.1)$$

where $X \in \mathfrak{X}(M)$, valid when both sides act on differential forms. Show that the right hand side obeys the same rule when acting on $\alpha \wedge \beta$ as does L_X in part (a).

(c) Show that the Cartan formula (8.1) is valid when acting on 0-forms and 1-forms.

(d) Explain why parts (a)–(c) prove the Cartan formula in all cases (that is, when acting on arbitrary differential forms).

3. (DTB) A useful formula, given in class and found in the book, is

$$\begin{aligned} d\omega(X_0, \dots, X_r) &= \sum_{i=0}^r (-1)^i X_i \omega(X_0, \dots, \hat{X}_i, \dots, X_r) \\ &\quad + \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_r), \end{aligned} \quad (8.2)$$

where $\omega \in \Omega^r(M)$, $X_0, \dots, X_r \in \mathfrak{X}(M)$ and where the hat means to omit the indicated vector field.

(a) Use the Cartan formula (8.1) to prove (8.2) in the cases $r = 0$ and $r = 1$. Note that when acting on a 0-form, i_X is zero. You will need to use the fact that the Lie derivative is a derivation.

(b) Use the Cartan formula and induction to prove (8.2) in the general case of arbitrary r .