Physics 250
Fall 2015
Homework 12
Due Friday, November 20, 2015

Reading Assignment: Lecture notes for Monday, Nov. 9 and Friday, Nov. 13. The topics covered in lecture this week were connection, torsion and curvature. I tried to cover these from three standpoints: First, the intuitive one, based on infinitesimals and coordinate-based calculations; second, the rigorous approach based on the covariant derivative operator ∇; and third, the one based on Cartan's method of converting everything possible into the language of differential forms, and introducing the idea of tensor-valued differential forms and the covariant exterior derivative D.

As I have explained, my approach to this topic is to introduce the connection first, and cover the topics that depend only on a connection, before introducing a metric. Nakahara, on the other hand, kind of mixes the subjects together. So to follow the flow of logic in the lectures it is necessary to skip around a bit in the book.

With this in mind, please read Nakahara, pp. 247–253 and 255–260, that is, everything about connections but skipping the parts that pertain to a metric. Also, in lecture I presented Cartan's approach to torsion and curvature, which is covered by Nakahara in pp. 283–286. Nakahara introduces non-coordinate bases late in the discussion, and when he does, they are orthonormal bases. But an orthonormal basis is only meaningful if you have a metric, and all of Cartan's formalism for torsion and curvature is independent of a metric (and thus works in arbitrary bases, coordinate or non-coordinate). I have introduced non-coordinate bases at the beginning because it is much too restrictive and counter to the geometrial philosophy to be chained to coordinate bases, and because the covariant exterior derivative is such a powerful tool.

We will take up metrics this week.

1. (DTB) Let $\{e_\mu\}$ be an arbitrary basis of vector fields (not necessarily a coordinate basis) and let $\{\theta^\mu\}$ be the dual basis of 1-forms. Define the connection 1-form (really a matrix of 1-forms),

$$\omega^\mu{}_\nu = \Gamma^\mu_{\alpha\nu}\,\theta^\alpha, \tag{12.1}$$

the torsion 2-form (a vector-valued 2-form),

$$T^\mu = d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu, \tag{12.2}$$

and the curvature 2-form (tensor-valued 2-form),

$$R^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\sigma \wedge \omega^\sigma{}_\nu. \tag{12.3}$$

Although we loosely talk about T^μ as the torsion tensor, really for fixed value of μ this is a 2-form whose value on a pair of vectors is one component of the vector into which those two vectors are mapped. The torsion tensor itself can be written,

$$T = e_\mu \otimes T^\mu, \tag{12.4}$$

and similarly the curvature is

$$R = e_\mu \otimes \theta^\nu \otimes R^\mu{}_\nu. \tag{12.5}$$

The expression (12.4) gives the torsion in terms of its components with respect to the given basis. To believe that the definition has a geometrical meaning, independent of basis, we should show that the answer does not change when we change basis.

Let $M^\mu{}_\nu(x)$ be a field of nonsingular matrices, that map one basis into another according to

$$e'_\mu = e_\nu \, M^\nu{}_\mu. \tag{12.6}$$

It follows from this that the dual basis transforms according to

$$\theta'^\mu = (M^{-1})^\mu{}_\nu \, \theta^\nu. \tag{12.7}$$

Like the fields of frames (primed and unprimed), the field of matrices $M^\mu{}_\nu$ is in general only defined locally.

(a) Find $\omega'^\mu{}_\nu$ in terms of $M^\mu{}_\nu$ and $\omega^\mu{}_\nu$, that is, find the transformation law for the connection 1-forms under a change of basis.

Comments and Hint: In class we derived the transformation law for the connection coefficients under a change of coordinates,

$$\Gamma'^\sigma_{\mu\nu} = \frac{\partial x'^\sigma}{\partial x^\gamma}\Big(\frac{\partial^2 x^\gamma}{\partial x'^\mu \partial x'^\nu} + \frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial x^\alpha}{\partial x'^\mu}\,\Gamma^\gamma_{\alpha\beta}\Big), \tag{12.8}$$

but I recommend that you do not try to use this in finding the transformation law of $\omega^\mu{}_\nu$. The reason is that (12.8) gives the transformation law of Γ under a coordinate transformation, while in this problem we are interested in more general, non-coordinate bases. However, once you have your answer, you should be able to check that (12.8) is a special case.

I suggest you approach this problem by starting with the definition of the connection coefficients,

$$\nabla_\alpha e_\beta = \Gamma^\gamma_{\alpha\beta}\, e_\gamma, \qquad \nabla'_\mu e'_\nu = \Gamma'^\sigma_{\mu\nu}\, e'_\sigma, \tag{12.9}$$

where $\nabla_\alpha = \nabla_{e_\alpha}$ and $\nabla'_\mu = \nabla_{e'_\mu}$. Then, in terms of the connection 1-forms,

$$\nabla e_\alpha = e_\beta \otimes \omega^\beta{}_\alpha, \qquad \nabla e'_\mu = e'_\nu \otimes \omega'^\nu{}_\mu. \tag{12.10}$$

and then use (12.6) and (12.7) to solve for $\omega'^\mu{}_\nu$. Use the notation

$$e_\mu(f) = f_{,\mu}, \tag{12.11}$$

and note that

$$df = f_{,\mu}\, \theta^\mu. \tag{12.12}$$

(b) Now show that (12.4) is independent of the basis. It is part of the definition of the D operator (covariant exterior derivative) that when it acts on a tensor it produces another tensor.

2. (DTB) The most important results that follow from the definitions (12.4) and (12.5) are

$$(DT)^\mu = dT^\mu + \omega^\mu{}_\nu \wedge T^\nu = R^\mu{}_\nu \wedge \theta^\nu, \tag{12.13}$$

and

$$(DR)^\mu{}_\nu = dR^\mu{}_\nu + \omega^\mu{}_\sigma \wedge R^\sigma{}_\nu - \omega^\sigma{}_\nu \wedge R^\mu{}_\sigma = 0. \tag{12.14}$$

These were proven in class, and in the notes. These are the Cartan versions of the first and second Bianchi identities.

The usual statement of the first Bianchi identity is this: If the torsion vanishes, then

$$R^\mu{}_{[\nu\alpha\beta]} = 0, \tag{12.15}$$

where the square brackets [. . .] mean to completely antisymmetrize in the enclosed indices. This was shown in class.

The usual statement of the second Bianchi identity is this: If the torsion vanishes, then

$$R^\mu{}_{\nu[\alpha\beta;\sigma]} = 0. \tag{12.16}$$

Here the semicolon is the usual notation in the GR literature for the components of the covariant derivative, that is,

$$R^\mu{}_{\nu\alpha\beta;\sigma} = (\nabla_\sigma\, R)^\mu{}_{\nu\alpha\beta}. \tag{12.17}$$

Starting from (12.14) (which notice is true whether or not the torsion vanishes), show that (12.16) holds when the torsion vanishes.