

1) Nakahara exercise 5.3:

Let $f: M \rightarrow N$, $g: N \rightarrow P$ then $g \circ f: M \rightarrow P$ and

$$(g \circ f)_* = g_* \circ f_*$$

Let the pushforward of V act on a function on P , call it h , then

$$\begin{aligned} ((g \circ f)_* V)[h] &= V[h \circ g \circ f] \\ &= (f_* V)[h \circ g] \\ &= (g_* (f_* V))[h] \\ &= (g_* \circ f_* V)[h] \quad \text{for all } h. \end{aligned}$$

Hence, $(g \circ f)_* = g_* \circ f_*$ as was to be shown.

Nakahara 5.5: Let f, g, V be as above and ω a one form on P . Then $(\omega \in T_{g(f(p))}^* P)$

$$\begin{aligned} \langle (g \circ f)^* \omega, V \rangle &= \langle \omega, (g \circ f)_* V \rangle \quad (V \in T_p M) \\ &= \langle \omega, (g_* \circ f_*) V \rangle \\ &= \langle g^* \omega, f_* V \rangle \\ &= \langle f^* \circ g^* \omega, V \rangle \quad \text{for all } \omega, V \end{aligned}$$

Then

$$(g \circ f)^* = f^* \circ g^* \quad \text{as was to be shown.}$$

2) We'd like to show that $\gamma = f \circ \alpha$ is an integral curve of $(f_* X)$ when α is an integral curve of X .

Well,

$$\frac{d\tau^i}{dt} = \frac{d}{dt} (f\circ\sigma)^i = \frac{\partial f^i}{\partial x^j} \frac{d\sigma^j}{dt}$$

Meanwhile,

$$(f_* X)[h] = X[h \circ f] = X^i \frac{\partial}{\partial x^i}(h \circ f) = X^i \frac{\partial h}{\partial y^j} \frac{\partial f^j}{\partial x^i}$$

so,

$$(f_* X)^j = X^i \frac{\partial f^j}{\partial x^i}$$

From the fact that σ is an integral curve of X
we have,

$$X^i = \frac{d\sigma^i}{dt} \quad \text{i.e. } X = \frac{d\sigma^i}{dt} \frac{\partial}{\partial x^i}$$

Then

$$(f_* X)^j = \frac{d\sigma^i}{dt} \frac{\partial f^j}{\partial x^i} = \frac{d\tau^j}{dt} \quad \checkmark$$

as was to be shown.

3) The discussion on pg 195 of Nakahara
shows that we can expand the lie derivative
of a tensor as follows, (let $\omega \in \mathcal{X}^*(M)$, $\gamma \in \mathcal{X}(M)$),

$$\begin{aligned} \mathcal{L}_X \delta(\omega, \gamma) &= X(\delta(\omega, \gamma)) - \delta(\mathcal{L}_X \omega, \gamma) - \delta(\omega, \mathcal{L}_X \gamma) \\ &= X(\omega(\gamma)) - (\mathcal{L}_X \omega)(\gamma) - \omega(\mathcal{L}_X \gamma) \\ &= X(\omega(\gamma)) - (X(\omega(\gamma)) - \omega(\mathcal{L}_X \gamma)) - \omega(\mathcal{L}_X \gamma) \\ &= 0 \end{aligned}$$

as was to be shown!

4) a) First suppose $e_\mu = \frac{\partial}{\partial y^\mu}$ then,

$$[e_\mu, e_\nu]f = \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y^\nu} f - \frac{\partial}{\partial y^\nu} \frac{\partial}{\partial y^\mu} f = 0$$

Conversely suppose $[e_\mu, e_\nu] = 0$, choose an arbitrary point p of M and a neighborhood U of p .

Starting with arbitrary coordinates $x^i|_{U(p)}$ around p we extend them using the map $x: \mathbb{R}^n \rightarrow M$ by

$$\begin{aligned} x(y^1, \dots, y^n) &= e^{y^\mu e_\mu} x^i|_p \\ &= e^{y^1 e_1} \dots e^{y^n e_n} x^i|_p \end{aligned}$$

This map is locally invertible as long as its Jacobian has non-zero determinant. The Jacobian is

$$\begin{aligned} \frac{\partial x^i}{\partial y^\nu} &= e^{y^1 e_1} \dots e_\nu e^{y^n e_n} \dots e^{y^\mu e_\mu} x^i|_p \\ &= e_\nu x^i \quad \leftarrow \text{commutativity of } e_\nu \end{aligned}$$

i.e. it's the matrix whose columns are the e_ν , i.e. the e_ν are by assumption linearly independent and so the map is locally invertible giving scalar functions $y^\nu: M \rightarrow \mathbb{R} \quad \nu=1, \dots, n$.

The above argument shows further that the vector fields generated by these functions are precisely

$$\frac{\partial}{\partial y^\nu} = e_\nu,$$

as was to be shown.

4) b) I'll use greek indices from the beginning of the alphabet for the $\frac{\partial}{\partial x^\alpha}$ basis and later ones for the e_μ basis. Define matrices e_μ^α and e^μ_α that are

$$e_\mu = e_\mu^\alpha \frac{\partial}{\partial x^\alpha} \quad \text{and} \quad \frac{\partial}{\partial x^\alpha} = e^\mu_\alpha e_\mu$$

these are clearly inverses of one another.
We compute

$$[e_\mu, e_\nu] f = C_{\mu\nu}^\sigma e_\sigma f$$

$$\begin{aligned} &= [e_\mu e_\nu - e_\nu e_\mu] f \quad \text{contain } x \text{ dependance} \\ &= \left[e_\mu^\alpha \frac{\partial}{\partial x^\alpha} e_\nu^\beta \frac{\partial}{\partial x^\beta} - e_\nu^\beta \frac{\partial}{\partial x^\beta} e_\mu^\alpha \frac{\partial}{\partial x^\alpha} \right] f \\ &= e_\mu^\alpha \left(\frac{\partial e_\nu^\beta}{\partial x^\alpha} \right) \frac{\partial f}{\partial x^\beta} + e_\mu^\alpha e_\nu^\beta \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} - e_\nu^\beta \left(\frac{\partial e_\mu^\alpha}{\partial x^\beta} \right) \frac{\partial f}{\partial x^\alpha} \\ &\quad - e_\nu^\beta e_\mu^\alpha \frac{\partial^2 f}{\partial x^\beta \partial x^\alpha} \\ &= e_\mu^\alpha \frac{\partial e_\nu^\beta}{\partial x^\alpha} \frac{\partial f}{\partial x^\beta} - e_\nu^\beta \left(\frac{\partial e_\mu^\alpha}{\partial x^\beta} \right) \frac{\partial f}{\partial x^\alpha} \end{aligned}$$

$$\Rightarrow C_{\mu\nu}^\sigma = e_\mu^\alpha \left(e_\nu^\beta \frac{\partial e_\nu^\alpha}{\partial x^\beta} - e_\nu^\beta \frac{\partial e_\mu^\alpha}{\partial x^\beta} \right).$$

From $\dot{x}^\alpha \frac{\partial}{\partial x^\alpha} = v^\mu e_\mu$ we get $\dot{x}^\alpha = e_\mu^\alpha v^\mu$

Then,

$$L(x^\alpha, \dot{x}^\alpha) = \bar{L}(x^\alpha, \cancel{v^\mu}) = \bar{L}(x^\alpha, e_\beta^\alpha \dot{x}^\beta)$$

and

$$\begin{aligned} P_\alpha &= \frac{\partial \bar{L}}{\partial \dot{x}^\alpha} = \frac{\partial \bar{L}}{\partial v^\mu} \frac{\partial}{\partial \dot{x}^\alpha} (e_\beta^\alpha \dot{x}^\beta) = e_\alpha^\mu \cancel{\frac{\partial \bar{L}}{\partial v^\mu}} \\ &= e_\alpha^\mu \cancel{\Pi}_\mu \end{aligned}$$

The equations of motion are,

$$\begin{aligned}\frac{d\pi_\alpha}{dt} &= \frac{\partial L}{\partial x^\alpha} = \frac{\partial L}{\partial x^\alpha} + \frac{\partial L}{\partial v^\mu} \frac{\partial}{\partial x^\alpha} (e^\mu_\beta \dot{x}^\beta) \\ &= \frac{\partial L}{\partial x^\alpha} + \frac{\partial L}{\partial v^\mu} \frac{\partial e^\mu_\beta}{\partial x^\alpha} e^\beta_\nu v^\nu\end{aligned}$$

Meanwhile, $\pi_\mu = e^\alpha_\mu p_\alpha$ implies that,

$$\begin{aligned}\frac{d\pi_\mu}{dt} &= \frac{de^\alpha_\mu}{dt} p_\alpha + e^\alpha_\mu \frac{dp_\alpha}{dt} \\ &= \frac{\partial e^\alpha_\mu}{\partial x^\beta} \dot{x}^\beta p_\alpha + e^\alpha_\mu \frac{dp_\alpha}{dt} = \frac{\partial e^\alpha_\mu}{\partial x^\beta} e^\beta_\nu v^\nu e^\sigma_\alpha \pi_\sigma + e^\alpha_\mu \frac{dp_\alpha}{dt} \\ &= e^\alpha_\mu \left(\frac{\partial L}{\partial x^\alpha} + \pi_\nu \frac{\partial e^\alpha_\nu}{\partial x^\mu} e^\mu_\nu v^\nu \right) + \frac{\partial e^\alpha_\mu}{\partial x^\beta} e^\beta_\nu v^\nu e^\sigma_\alpha \pi_\sigma\end{aligned}$$

Apologies, the notation is terrible, if $e^\mu_\alpha = e^\alpha_\mu = \frac{\partial}{\partial x^\mu}$ call
 ~~$(e^{-1})^\mu_\alpha = (e^{-1})^\alpha_\mu = \frac{\partial}{\partial x^\alpha}$~~ and
 $e^\mu_\alpha = (e^{-1})^\mu_\alpha$ then we have ~~$(e^{-1})^\mu_\alpha = (e^{-1})^\alpha_\mu = \frac{\partial}{\partial x^\alpha}$~~

$$\partial_x (e^{-1})^\mu_\alpha \cdot e^\mu_\nu = - (e^{-1})^\mu_\alpha \partial_x e^\mu_\nu \text{ so, then,}$$

$$\begin{aligned}\frac{d\pi_\mu}{dt} &= e^\alpha_\mu \frac{\partial L}{\partial x^\alpha} + \left(e^\beta_\nu v^\nu (e^{-1})^\sigma_\alpha \pi_\sigma \frac{\partial e^\alpha_\mu}{\partial x^\sigma} + \pi_\nu v^\nu \frac{\partial (e^{-1})^\sigma}{\partial x^\mu} e^\mu_\nu e^\alpha_\sigma \right) \\ &= e^\alpha_\mu \frac{\partial L}{\partial x^\alpha} + \left(v^\nu \pi_\sigma e^\sigma_\nu (e^{-1})^\sigma_\mu \frac{\partial e^\alpha_\mu}{\partial x^\sigma} - \pi_\nu v^\nu (e^{-1})^\sigma_\mu \frac{\partial e^\nu_\sigma}{\partial x^\mu} e^\alpha_\sigma \right) \\ &= e^\alpha_\mu \frac{\partial L}{\partial x^\alpha} + (e^{-1})^\sigma_\mu \pi_\sigma v^\sigma \left(e^\beta_\nu \frac{\partial e^\alpha_\mu}{\partial x^\beta} - \frac{\partial e^\nu_\sigma}{\partial x^\mu} e^\alpha_\nu \right)\end{aligned}$$

$$\Rightarrow \boxed{\frac{d\pi_\mu}{dt} = e^\alpha_\mu \frac{\partial L}{\partial x^\alpha} + C^\sigma_\mu \pi_\sigma v^\sigma}$$