

1a) Recall the left coset of H is $[g] = \{gh \mid h \in H\}$ and similarly for right cosets. Given a coset gH we can obtain any other left coset by the action $L_{g'g^{-1}}$, $gH \mapsto g'H$. This is a bijection on G and clearly restricts to a bijection between gH and $g'H$. Then

$$\#gH = \#g'H = \#eH = \#H$$

Equivalence relations break a set into a union of disjoint subsets and so we have

$$\#G = \#H \# \left(\frac{G}{H} \right) \Rightarrow \# \left(\frac{G}{H} \right) = \frac{\#G}{\#H}$$

b) Associativity and existence of an identity are immediately inherited from G : e.g. $\mathbb{I}_e x = x \Rightarrow e \in \mathbb{I}_x$. We need closure, suppose $g, h \in \mathbb{I}_x$ then (taking \mathbb{I}_g to be a left action, a similar argument works for right actions)

$$\mathbb{I}_g \mathbb{I}_h x = x = \mathbb{I}_{gh} x \Rightarrow gh \in \mathbb{I}_x$$

We also need inverses: suppose $g \in \mathbb{I}_x$

$$\mathbb{I}_g x = x \Rightarrow \mathbb{I}_{g^{-1}} \mathbb{I}_g x = \mathbb{I}_{g^{-1}} x \Rightarrow \mathbb{I}_{g^{-1}} x = x$$

and indeed $g^{-1} \in \mathbb{I}_x$. Now, suppose $y \in [x]$ so that we have $y = gx$, we'd like to label y by $g\mathbb{I}_x$ but is this unique? Suppose further $y = g'x$ then $gx = g'x \Rightarrow g^{-1}g'x = x \Rightarrow g^{-1}g' \in \mathbb{I}_x$ so $g^{-1}g'\mathbb{I}_x = \mathbb{I}_x \Rightarrow g'\mathbb{I}_x = g\mathbb{I}_x$ and indeed $g\mathbb{I}_x$ serves as a unique label. Then

$$\#[x] = \#(G/\mathbb{I}_x) = \#G / \#\mathbb{I}_x$$

1c) We'd like to show that the map $\Phi: F_\alpha \rightarrow F_\alpha$ depends only on $[\alpha]$ not on α alone. Consider two curves $\alpha, \alpha' \in [\alpha]$ then there is a homotopy between them $H(s, t)$ s.t. $H(0, t) = \alpha(t)$ and $H(1, t) = \alpha'(t)$. Just as we can uniquely lift the curves given an $\bar{x}_0 \in P^{-1}(x_0)$ we can uniquely lift the homotopy $\bar{H}(s, t)$. The lifted homotopy provides a homotopy between $\bar{\alpha}(t)$ and $\bar{\alpha}'(t)$ with $\bar{H}(0, t) = \bar{\alpha}(t)$ and $\bar{H}(1, t) = \bar{\alpha}'(t)$ in particular $\bar{H}(s, 1) \equiv \bar{x}_s$ is the same for all s and consequently $\bar{\alpha}(1) = \bar{\alpha}'(1) = \bar{x}_s$ is uniquely specified by any $\alpha \in [\alpha]$ and $\Phi: F_\alpha \rightarrow F_\alpha, \bar{x}_0 \mapsto \bar{x}_s$ depends only on $[\alpha]$.

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Now, we'd like to find a group action. Let $\alpha_1 \in [\alpha_1] = g_1$ and $\alpha_2 \in [\alpha_2] = g_2$ then,

$$\Phi_{g_1} \circ \Phi_{g_2}(\bar{x}_0) = \Phi_{g_1}(\bar{\alpha}_2(1) \Big|_{\bar{x}_0}^{\bar{x}_0}) = \bar{\alpha}_1(1) \Big|_{\bar{\alpha}_2(1) \bar{x}_0}^{\bar{x}_0}$$

This is a curve that follows α_2 first then α_1 , namely

$$\beta \equiv \overline{\alpha_2 * \alpha_1} \Big|_{\bar{x}_0}$$

and

$$\Phi_{g_1} \circ \Phi_{g_2} = \Phi_{g_2 g_1}$$

which is not a left action.

But we know how to fix that, let $\bar{\Phi}: F_\alpha \rightarrow F_\alpha$

$$\bar{\Phi}_{g_1}(\bar{x}_0) = \bar{\Phi}_{g_1^{-1}}(\bar{x}_0) \text{ then}$$

$$\begin{aligned} \bar{\Phi}_{g_1} \circ \bar{\Phi}_{g_2}(\bar{x}_0) &= \bar{\Phi}_{g_1^{-1}} \circ \bar{\Phi}_{g_2^{-1}}(\bar{x}_0) = \bar{\Phi}_{g_2^{-1} g_1^{-1}}(\bar{x}_0) = \bar{\Phi}_{(g_1 g_2)^{-1}}(\bar{x}_0) \\ &= \bar{\Phi}_{g_1 g_2}(\bar{x}_0) \end{aligned}$$

a left group action.

1d) Suppose that \bar{M} is connected then for any $\bar{x}_0, \bar{x}_f \in F_0$ we can connect them with a path $\bar{\alpha}$. By projecting this path to M via p we have a path $\alpha = p(\bar{\alpha})$ that is a loop based at x_0 and whose equivalence class satisfies, if $y = [\alpha]$ then

$$\bar{\Phi}_y^{-1}(\bar{x}_0) = \bar{x}_f$$

and this construction works for all \bar{x}_0 and $\bar{x}_f \in F_0$ so F_0 consists of a single orbit.

To prove the converse we use the contrapositive.

Suppose \bar{M} is not connected and let \bar{M}_1 and \bar{M}_2 be two connected components of \bar{M} . As mentioned

in the notes (but not proven) $p(\bar{M}_1) = p(\bar{M}_2) = M$. We

can see this as follows: Suppose this weren't the case then there exists $x \in M$ such that $x \notin p(\bar{M}_1)$ but for

any $x' \in p(\bar{M}_1)$ we can take a path β in M

connecting x and x' (recall M is connected) which

lifts to a path $\bar{\beta}$ in \bar{M}_1 connecting $\bar{x}' \in p^{-1}(x')$ to $\bar{x} \in p^{-1}(x)$, so $x \in p(\bar{M}_1)$, a contradiction. ~~Now~~

Now, let $x_0 \in M$ and let $y, z \in F_0 \subset p^{-1}(x_0)$ satisfy

$$y \in \bar{M}_1 \quad \text{and} \quad z \in \bar{M}_2.$$

Then for all $[\alpha] \in \pi_1(M, x_0) = G$, $\bar{\Phi}_{[\alpha]}^{-1}(y) \in \bar{M}_1$ because

if $\bar{\alpha}(0) = y$, $\bar{\alpha}(1) \in \bar{M}_1$ by continuity of $\bar{\alpha}$. Then there does not exist $[\alpha] \in G$ such that $\bar{\Phi}_{[\alpha]}^{-1}(y) = z$, so $[y]$ is not equal to F_0 !. This completes the proof.

From part b)

$$\# [y] = \# F_0 = \frac{\# G}{\# I_y}$$

\Rightarrow

$$\# F_0 \text{ divides } \# G.$$