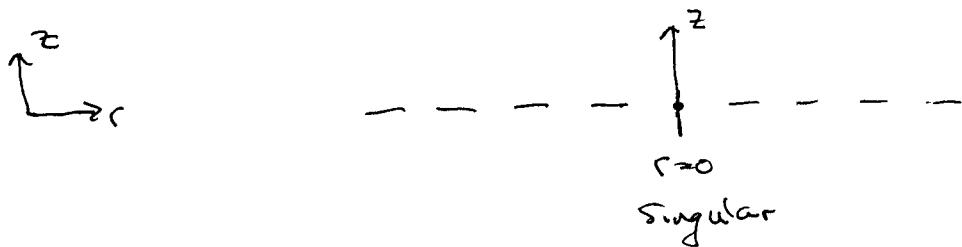


- 1) Figure 4.25 of Nakahara is a top down perspective of the line 'defect.' As the defect is cylindrically symmetric we can also take a cross section and visualize that. Taking z to be the vertical direction and r the radial cylindrical coordinate we have



This can be continuously deformed into the configuration



which is nonsingular at $r=0$ and asymptotically identical to the original configuration.

- 2) Recall that a deformation retract is a continuous map $H: X \times I \rightarrow I$ such that

$$H(x, 0) = x \quad \forall x \in X \quad H(x, 1) \in R \quad \text{where } R \subset X$$

is set you're retracting to.

$$H(x, t) = x \quad \forall x \in R \text{ and } \forall t.$$

Consider $\vec{v} \in \mathbb{R}^{n+1-\{0\}}$, we'd like to map \vec{v} to the unit vector pointing in the \vec{v} direction, $\hat{v} = \vec{v}/\|v\|$, $w = \|v\|$. We begin with $\vec{v} - t \vec{w}$ so that at $t=0$ we recover \vec{v} .

2) cont. At $t=1$ we want \hat{v} and so \vec{w} should \vec{v}^2/t cancel out \vec{v} and add in \hat{v} , we try

$$H(\vec{v}, t) = \vec{v} - t(\vec{v} - \hat{v})$$

Indeed $H(\vec{v}, 0) = \vec{v}$ $\forall \vec{v} \in \mathbb{R}^{n+1} - \{0\}$ and

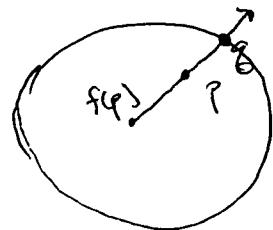
$$H(\vec{v}, 1) = \hat{v} \in S^n$$

Finally for $\vec{v} \in S^n$, $\|\vec{v}\|=1$ and so;

$$H(\hat{v}, t) = \hat{v} - t(\hat{v} - \hat{v}) = \hat{v} \quad \forall \hat{v} \in S^n.$$

This is the desired ~~retraction~~^{deformation}. The retraction is simply $r(\vec{v}) = \frac{\vec{v}}{\|\vec{v}\|}$ and satisfies $r \circ i = \text{id}_{S^n}$ where i is the inclusion $i: S^n \hookrightarrow \mathbb{R}^{n+1} - \{0\}$.

3) We proceed by contradiction. Suppose that $f: D^2 \rightarrow D^2$ has no fixed points then ~~there's~~ there's an auxiliary map given by connecting the image under f of p to p by a straightline and extending this line to the boundary:



This gives a mapping $\hat{f}: D^2 \rightarrow S^1$, $\hat{f}(p) = q$.

Now, \hat{f} shows ~~that~~ that D^2 and S^1 are of the same homotopy type because if we introduce the inclusion mapping $i: S^1 \hookrightarrow D^2$ sending a point of S^1 to the same point on the boundary $\cong S^1$ of D^2 , then

$$\hat{f} \circ i: S^1 \rightarrow S^1 \quad \hat{f} \circ i(p \in S^1) = \hat{f}(p) = p$$

\hat{f} acts as id on ∂D^2

3 cont.) So, $\hat{f} \circ i: S^1 \rightarrow S^1$ is the identity on S^1 .

Meanwhile, $i \circ \hat{f}: D^2 \rightarrow D^2$ is homotopic to the identity $i \circ \hat{f} \sim \text{id}_{D^2}$ because the disk D^2 is homotopic to any one of its points. Thus D^2 and S^1 are of the same homotopy type and

$$\pi_1(D) = \{c\} \cong \pi_1(S^1) = \mathbb{Z}$$

but this is clearly false. We conclude that f must have a fixed point.

4) a) We compute

$$\det U = 1 \Rightarrow ad - bc = 1$$

$$UU^T = 1 \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} |a|^2 + |b|^2 & ac^* + bd^* \\ a^*c + b^*d & |c|^2 + |d|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$U^T U = 1 \quad \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} |a|^2 + |c|^2 & a^*b + c^*d \\ ab^* + cd^* & |b|^2 + |d|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We have,

$$\boxed{|a|^2 + |b|^2 = 1}$$

Further $a^*c + b^*d = 0$

$$\Rightarrow a^*c + b^* \left(\frac{1+bc}{a} \right) = 0 \quad \text{from } \det U = 1 \text{ e.g.}$$

$$\Rightarrow \frac{(|a|^2 + |b|^2)c + b^*}{a} = 0 \quad \text{common denominator}$$

$$\Rightarrow \frac{c+b^*}{a} = 0 \quad \text{from } |a|^2 + |b|^2 = 1$$

$$\Rightarrow \boxed{c = -b^*}$$

Putting this back into the first equation

$$a^* (-b^*) + b^* d = 0 \Rightarrow \boxed{d = a^*}$$

Taking $a = x_0 - ix_3$ and $b = -x_2 - ix_1$, we have,

$$U = \begin{pmatrix} x_0 - ix_3 & -x_2 - ix_1 \\ x_2 - ix_1 & x_0 + ix_3 \end{pmatrix} = x_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \vec{x} \cdot \vec{\sigma}$$

det $U = |a|^2 + |b|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$. ✓ Conversely:
well, any point of the three sphere can be described
by $U = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ s.t. $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$ and
by setting $a = x_0 - ix_3$ and $b = -x_2 - ix_1$ in these
coordinates you get an $U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \in SU(2)$.

b) To show that P is a homomorphism we'd like to
show that when $U'' = U'U$ we have

$$P(U'') = P(U'U) = P(U')P(U)$$

$$\text{or } R''_{ik} = R'_{ij} R_{jk} \quad \text{w/ } R''_{ik} = P(U'') \quad = \frac{1}{2} \text{tr}(U'' \sigma_i U'' \sigma_j) \quad \text{etc.}$$

If we take $M = U^+ \sigma_i U$ in the given formula

$$M = \frac{1}{2} \text{tr}(M) + \frac{1}{2} \sum_{i=1}^3 \sigma_i \text{tr}(\sigma_i M)$$

we have because of cyclic tr and σ_i traceless

$$\begin{aligned} U^+ \sigma_i U &= \frac{1}{2} \text{tr}(U^+ \sigma_i U) + \frac{1}{2} \sum_{j=1}^3 \sigma_j \text{tr}(\sigma_j U^+ \sigma_i U) \\ &= \sum_{j=1}^3 \sigma_j R_{ij} \end{aligned}$$

4b)

Then,

$$\begin{aligned}
 R_{ik}'' &= \frac{1}{2} \text{tr} (U^T \sigma_i U \sigma_k) \\
 &= \frac{1}{2} \text{tr} \left(U^T \sum_{l=1}^3 R_{il} \sigma_l U \sigma_k \right) \\
 &= \sum_{l=1}^3 R_{il} \frac{1}{2} \text{tr} (U^T \sigma_l U \sigma_k) \\
 &= \sum_{l=1}^3 R_{il} R_{lk}
 \end{aligned}$$

as was to be shown. Because the pauli matrices satisfy

$$\sigma_i \sigma_j = S_{ij} \cdot \text{Id} + i \sum_k \epsilon_{ijk} \sigma_k$$

we have

$$R_{ij} = \frac{1}{2} \text{tr} (U^T \sigma_i U \sigma_j)$$

is the identity when $U = \pm \text{Id}$ because then the off diagonal components of R_{ij} vanish ($\text{tr} \sigma_i = 0$) and the diagonal components are 1s:

$$\boxed{\ker p = \{\text{Id}, -\text{Id}\} \cong \mathbb{Z}_2}$$

c) Well, $\Psi(t)$ must have norm equal to one, so

$$\begin{aligned}
 \langle \Psi(t) | \Psi(t) \rangle &= 1 \\
 \Rightarrow \langle \Psi_0 | U^T U | \Psi_0 \rangle &= 1
 \end{aligned}$$

which is satisfied if $U^T U = 1$, indeed the unitary nature of U is guaranteed as we see by direct solution of the Schrödinger equation for infinitesimal dt .

$$\begin{aligned}
 i \frac{d\Psi}{dt} &= \vec{\omega}(t) \cdot \frac{\vec{\sigma}}{2} \Psi \\
 \Rightarrow \Psi(t) &= e^{-i \vec{\omega}(t) \cdot \frac{\vec{\sigma}}{2}} \Psi(0)
 \end{aligned}$$

The exponential of a hermitian operator is unitary and P6/7
 the time development for finite times can be seen as
 the composition of unitaries which is a unitary.

Finally, because $U(0) = \mathbb{1} \Rightarrow \det U(0) = 1$
 and since $\vec{\omega}(t)$ is continuously connected to $\vec{\omega}(0)$
 we have $\det U(t) = 1$ for all t .

Similarly $d\vec{s} = \vec{\omega}(t) \times \vec{s} dt$

implies that the change in \vec{s} is both perpendicular
 to \vec{s} , and hence preserves its length, and also preserves
 the orientation of \vec{s} . This immediately implies that
 $R(t) \in SO(3)$.

As discussed above $U^\dagger \sigma_i U = R_{ij} \sigma_j$ (sum implied)

and so

$$\begin{aligned} S_i(t) &= \langle \psi(t) | \frac{t}{2} \sigma_i | \psi(t) \rangle \\ &= \langle \psi(0) | U^\dagger(t) \frac{t}{2} \sigma_i U(t) | \psi(0) \rangle \\ &= \langle \psi(0) | \frac{t}{2} R_{ij}(t) \sigma_j | \psi(0) \rangle \\ &= R_{ij}(t) \langle \psi(0) | \frac{t}{2} \sigma_j | \psi(0) \rangle \\ &= R_{ij}(t) S_j(0) \end{aligned}$$

and so indeed $R_{ij}(t) = \frac{t}{2} \text{tr}(\sigma_j U^\dagger(t) \sigma_i U(t))$.

5) In the class notes prof. Littlejohn has shown that
 K is injective, it remains to show that K is
 surjective and a homomorphism. Recall that X and Y
 are of the same homotopy type so we have maps
 $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that $g \circ f \sim id_Y$ and $f \circ g \sim id_X$.

5 cont.) To show surjectivity we need to show that for $\beta \in \pi_1(Y)$ every β there exists an α such that $f(\alpha) \in \beta$.

Choose $\beta \in \pi_1(Y)$ and consider $\gamma = g(\beta)$, a loop in X based at $x = g(y_0)$, given that β was based at y_0 . Now since $y_0 = f(x_0)$ we know that we can connect x and x_0 since $g \circ f \sim id_X$ we can use the same curves η introduced in the notes:

$$\alpha = \eta^{-1} \circ \gamma \circ \eta.$$

This α satisfies $f(\alpha) \in \beta$; ~~since~~ By assumption $f \circ g \sim id_Y$, and let N_t be the homotopy then

$$f(\alpha) = f(\eta^{-1}) \circ f \circ g(\beta) \circ f(\eta) \quad \text{by homomorphism (see below)}$$

$$f_t(\alpha) = f_t(\eta^{-1}) \circ (N_t \circ (\beta)) \circ f_t(\eta)$$

where $f_t(\eta) \circ (f \circ g)(s) = f(\eta)(t + (1-t)s)$ (similar to η_t from notes)

Then $f_t(\alpha)$ is a homotopy that shows $f(\alpha) \sim \beta$, hence $f(\alpha) \in \beta$ and such α can be found for every β ! Thus K is surjective.

K is also a homomorphism for

$$f(\alpha \circ \alpha') = f(\alpha) \circ f(\alpha').$$

$$\Rightarrow K(\alpha \circ \alpha') = K(\alpha) K(\alpha')$$

This completes the proof that

$$\pi_1(X, x_0) \cong \pi_1(Y, y_0)$$

when X and Y are of the same homotopy type.