

1) a) Let the generators of G be $x_1 = (1, 0)$ and $x_2 = (0, 1)$. We seek a change of basis to generators $y_1 = (1, 3)$ and $y_2 = (y_{2x}, y_{2y})$. Let the change of basis be denoted $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The requirement that $y_i = Mx_i$ gives us $a=1$, $b=3$. We also require that $\det M = \pm 1$, this is because the inverse M^{-1} must have integer entries and since the entries of M^{-1} are given by cofactors divided by the $\det M$ they are only guaranteed to be integers if $\det M = \pm 1$. Then,

$$\det M = d - 3b = \pm 1$$

Choosing $\det M = +1$ for definiteness (we might as well preserve orientation) gives

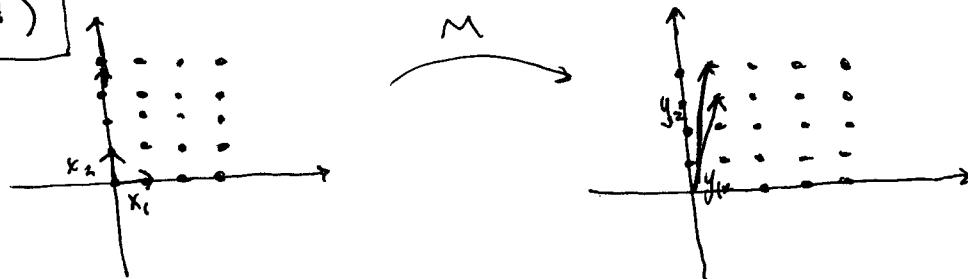
$$d = 1+3b$$

Then

$$\begin{pmatrix} y_{2x} \\ y_{2y} \end{pmatrix} = \begin{pmatrix} 1 & b \\ 3 & 1+3b \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ 1+3b \end{pmatrix}$$

Any choice of b works, so let's take $b=1$ then

$$y_2 = (1, 4)$$



1) b) Suppose that $M \in GL(r, \mathbb{Z})$ then as discussed in part a) $\det M = \pm 1$. Further suppose that M has a row or column whose entries share a common factor $a \neq \pm 1$. Denote by N the matrix which is identical to M except that the common factor of our supposed row or column has been cancelled out (e.g. $M = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$, $N = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$).

Then

$$\pm 1 = \det M = a \det N \Rightarrow \det N = \pm \frac{1}{a}$$

But N consists of only integer entries and cannot have a fractional determinant so this is a contradiction! We conclude that none of the entries of a row or column of M can share a common factor $a \neq \pm 1$ if $M \in GL(r, \mathbb{Z})$.

2) Let's build some intuition with the base cases, consider $h=1, g=1$, we have



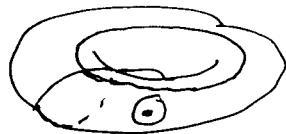
This surface is connected and as this will hold for all h and g , we have,

$$H_0(T^2_{h,g}) = \mathbb{Z}.$$

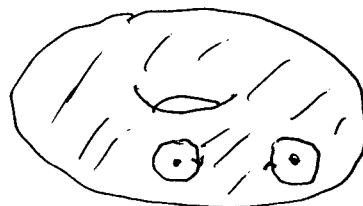
For $g \neq 0$ there are always boundaries due to the holes, and as there are no 3-simplices the operator ∂_3 vanishes hence

$$H_2(T^2_{h,g}) = \begin{cases} \mathbb{Z}^3 & g \neq 0 \\ \mathbb{Z} & g = 0. \end{cases}$$

The interesting group is $H_1(T_{h,g}^2)$. For $h=1, g=1$ P3/5 we have 1 additional cycle of interest, the one which encircles the hole but this is a boundary of the rest of the torus. So



the 1-cycles which are not boundaries are just the usual generators: $H_1(T_{1,1}^2) = \mathbb{Z}^2$. For $h=1, g=2$ something interesting happens, now encircling one of the holes is not a boundary but encircling each in the proper orientation is a boundary of the ~~rest~~ rest of the torus.



This generalizes, we always have the $2h$ generators of the handles then there are g cycles for the holes but the appropriate linear combination of these that creates a boundary of the rest of the torus needs to be removed. So,

$$H_1(T_{h,g}^2) = \mathbb{Z}^{2h+g-1}$$

The Euler characteristic is,

$$\chi = \sum_{r=0}^n (-1)^r b_r(T_{h,g}^2) = 1 - (2h+g-1) + 0 \quad g \neq 0$$

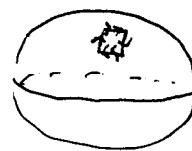
$$\Rightarrow \boxed{\chi = 2 - 2h - g} \quad g \neq 0$$

For

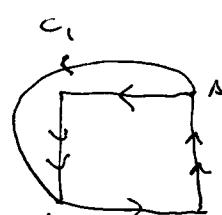
$$\boxed{\chi = 1 - 2h + 1 = 2 - 2h} \quad g = 0$$

3) Again consider the case $g=1$, a single cross cap on the sphere

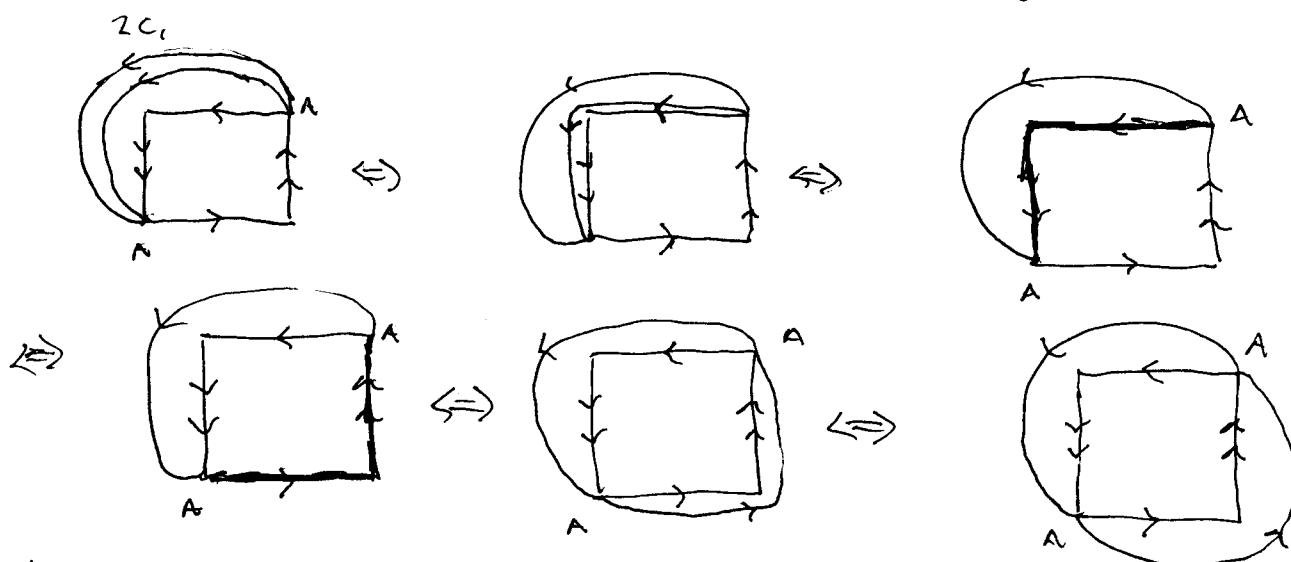
PH/5



The fundamental fact about cross caps is that you can create cycles that do not enclose them but whose even multiples do; let c_1 be the following cycle



Then $2c_1$ can be deformed as follows:



which encloses the cross cap.

Once again the sphere with g cross caps is connected and so,

$$H_0(S^2_g) = \mathbb{Z}$$

Because triangulations of surfaces with cross caps cannot internalize the cross cap boundaries we have both $H_2(S^2_g) \cong \mathbb{Z}_2(S^2_g) \cong \{\pm 0\}$ and so

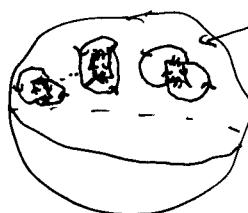
$$H_2(S^2_g) = \begin{cases} \{\pm 0\} & g \neq 0 \\ \mathbb{Z} & g = 0 \end{cases}$$

PS/5

Finally, for a sphere with g cross caps there are g cycles like c_i , above which are not boundaries. However, a linear combination with appropriate signs of $2c_i$,

$$\sum_{i=1}^g \text{sgn}(i) 2c_i$$

is a boundary



These bound remainders of sphere.

So we have,

$$H_1(S_g^2) = \begin{cases} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}^{g-1} \times \mathbb{Z}_2 & g \neq 0 \\ \{0\} & g = 0 \end{cases}$$

Unsurprisingly we have a non-zero torsion.
The Euler character is

$\chi = 1 - (g-1) + 0 = 2-g$	$g \neq 0$
$\chi = 1 - 0 + 1 = 2$	$g = 0$

Remark : If you added h holes to this analysis you'd have

$$H_1(S_{g,h}^2) = \mathbb{Z}^{h+g-1} \times \mathbb{Z}_2 \quad \text{and} \quad \chi = 2-h-g.$$

With this addition you've found the Euler characteristic for all compact 2d manifolds with and without boundaries!