Physics 250

Homework #2

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1) For $V$ a vector space and $U \subset V$ a vector subspace we define a relation

$$v_1 \sim v_2 \quad \text{if} \quad v_1 - v_2 \in U$$

This is an equivalence relation:

reflexive: $v_1 - v_1 = 0$ and $0 \in U$ because $U$ is a vector subspace. Then

$$v_1 \sim v_1.$$

Symmetric: If $v_1 - v_2 \in U$, say $v_1 - v_2 = u_1$ then

$v_2 - v_1 = -u_1$ and $-u_1 \in U$ by assumption. So

$$v_1 \sim v_2 \iff v_2 \sim v_1$$

Transitive: Given $v_1 \sim v_2$ and $v_2 \sim v_3$ we have

$v_1 - v_2 = u_1$, $v_2 - v_3 = u_2$ so

$$(v_1 - v_2) + (v_2 - v_3) = u_1 + u_2$$

$$\Rightarrow \quad v_1 - v_3 = u_3 \equiv u_1 + u_2 \quad \text{by assumption } u_3 \in U$$

So

$$v_1 \sim v_3.$$

We can give $V/U$ the structure of a vector space by defining

$$[v] + [w] = [v + w]$$

$$\alpha [v] = [\alpha v]$$

We need to show that these definitions are independent of representative. Suppose $v' = v + u_1$, $w' = w + u_2$ then

$$[v'] + [w'] = [v + u_1 + w + u_2] = [v + w] = [v] + [w]. \checkmark$$

Also

$$\alpha[v'] = [\alpha v + \alpha u_1] = [\alpha v] = \alpha[v]. \checkmark$$

Define the map $\hat{f} : V/_{\ker f} \to \operatorname{im} f$ by

$$\hat{f}([v]) = f(v)$$

We show that this is an isomorphism by showing that $\ker \hat{f} = \{\vec{0}\}$ and that $\hat{f}$ is a homomorphism. By assumption $u \in U = \ker f$ and we have $[u] = [0]$, so indeed $\hat{f}([0]) = 0$ and for any other equivalence class we have $f(v) \neq 0$ if $[v] \neq [0]$ so indeed $\ker \hat{f} = \{\vec{0}\}$. Homomorphism: $\hat{f}([v+w]) = \hat{f}([v]+[w])$

$$= f(v+w) = f(v) + f(w) = \hat{f}([v]) + \hat{f}([w]) \checkmark$$

To show that, $U^\perp$ = the space perpendicular to the kernel of $f$, is isomorphic to $V/\ker f$ we can show they have the same dimension.

From our toy index theorem

$$\underbrace{\dim \ker \hat{f}}_{0 \text{ from above}} + \dim \operatorname{im} \hat{f} = \dim\left(V/_{\ker f}\right)$$

$$\Rightarrow \dim \operatorname{im} \hat{f} = \dim \operatorname{im} f = \dim\left(V/_{\ker f}\right)$$

$$\Rightarrow \dim\left(V/_{\ker f}\right) = \dim \operatorname{im} f = \dim V - \dim \ker f = \dim V - \dim U$$

On the other hand it's clear that

$$\dim(U^\perp) = \dim V - \dim U$$

so indeed,

$$\dim\left(V/_{\ker f}\right) = \dim(U^\perp)$$

To specify the isomorphism simply specify how to map basis vectors in one space onto those of the other.

2) a) Consider the map $r : V^* \to U^*$ which is given by restriction, $r(\alpha) = \alpha|_U$. Again our linear index theorem gives

$$\dim \ker r + \dim \operatorname{img} r = \dim V^* = \dim V$$

$$\dim X^* + \dim U^* = \dim V$$

by assumption that $\alpha \in X^*$ vanishes on $U$.

Then,

$$\boxed{\dim X^* + \dim U = \dim V}$$

b) Well we know that the $\ker f$ is a linear subspace of $V$, what can we say about its annihilator space? Well the annihilator space should be a linear subspace of $V^*$, so let's guess that it's $\operatorname{img} f^*$ and check if this is true. Let $\ker f = U$ and $X^*$ be $X^* = \{\alpha \in V^* \mid \alpha(u) = 0 \ \forall u \in U\}$

then for $\beta \in \operatorname{img} f^* \subset V^*$ we have $\beta = f^*(\gamma)$ for some $\gamma \in W^*$

and

$$\beta(v) = (f^*\gamma)(v) = \gamma(f(v)) = 0 \quad \forall\, v \in U \text{ since } f(v) = 0$$

for these vectors, so $\operatorname{img} f^* \subset X^*$.

Now, suppose instead $\beta \in X^*$ and define $\gamma \in W^*$ by $\gamma(w) = \beta(v)$ with $f(v) = w$. This definition is independent of $v$ because if we suppose $f(v) = f(v') = w$ then $f(v - v') = 0$ and $v - v' \in \ker f$. But then

$$\beta(v') = \beta(v + v' - v) = \beta(v) \quad \text{because } \beta \in X^*$$

Finally we have $\beta = f^*\gamma$ and so $X^* \subset \operatorname{img} f^*$

Then $\operatorname{Img} f^* = X^*$ and from part a) we have,

$$\dim \ker f + \dim \operatorname{img} f = \dim V$$

$$\Rightarrow \boxed{\dim \operatorname{img} f = \dim V - \dim \ker f = \dim X^* + \dim \ker f - \dim \ker f = \dim \operatorname{img} f^*}$$

3) We can just check directly that (4) satisfies (5):

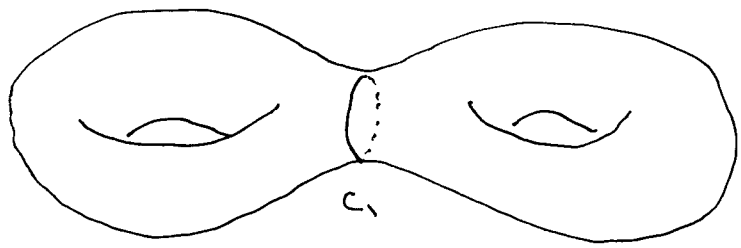
$$\langle \tilde{f} w, v \rangle_g = \langle (g^{-1} f^* G) w, v \rangle_g$$

$$= ((f^* G) w, v) \quad \text{standard pairing}$$

$$= (G(w), f(v)) \quad \text{def. of } f^*$$

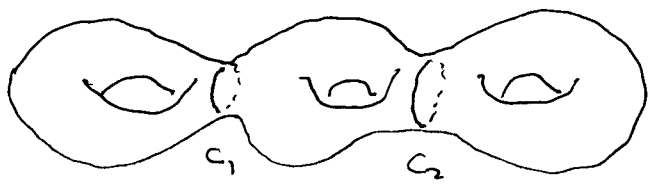
$$= \langle w, f v \rangle_G \quad \checkmark$$

4) a)




The ~~cycle~~ cycle $c_1$ is the boundary of the left or right halves but is not continuously deformable to a point.

b)




The cycles $c_1$ and $c_2$ are homologous as they bound the middle handle but cannot be deformed into one another.