1) For \( V \) a vector space and \( U \subset V \) a vector subspace we define a relation

\[ u_1 \sim u_2 \text{ if } u_1 - u_2 \in U \]

This is an equivalence relation:

reflexive: \( u_1 - u_1 = 0 \) and \( 0 \in U \) because \( U \) is a vector subspace. Then

\[ u_1 \sim u_1. \]

symmetric: If \( u_1 - u_2 \in U \), say \( u_1 - u_2 = u_3 \) then

\[ \begin{align*}
    u_2 - u_1 &= -u_3 \\
    -u_3 &\in U \text{ by assumption. So}
\end{align*} \]

\[ u_1 \sim u_2 \iff u_2 \sim u_1. \]

transitive: Given \( u_1 \sim u_2 \) and \( u_2 \sim u_3 \) we have

\[ u_1 - u_2 = u_3, \quad u_2 - u_3 = u_4 \]

so

\[ (u_1 - u_2) + (u_2 - u_3) = u_1 + u_4 \]

\[ u_1 - u_3 = u_4 \in U \text{ by assumption } u_3 \in U \]

so

\[ u_1 \sim u_3. \]

We can give \( V/U \) the structure of a vector space by defining

\[ [u] + [w] = [u + w] \]

or

\[ [wu] = [w][u] \]

We need to show that these definitions are independent of representative. Suppose \( u' = u + u_1, \ w' = w + u_2 \) then

\[ [u'] + [w'] = [u + u_1 + w + u_2] = [u + w] = [u] + [w]. \]
Also, $\alpha [U] = [\alpha u + \alpha u] = [\alpha u] = \alpha [u]$. \\

Define the map $\hat{f} : \mathbb{V}/\ker f \rightarrow \text{im} f$ by $\hat{f}([U]) = f(u)$.

We show that this is an isomorphism by showing that $\ker \hat{f} = \mathbb{E}$. By assumption $U \cap \ker f = \ker f$ and we have $[u] = [0]$, so indeed $\hat{f}(0) = 0$ and for any other equivalence class we have $\hat{f}(u) \neq 0$ if $[u] \neq [0]$. So indeed $\ker \hat{f} = \mathbb{E}$.

Homomorphism: $\hat{f}([u] + [w]) = \hat{f}([u]) + \hat{f}([w]) = f(u) + f(w) = \hat{f}(0)$.

To show that $U^\perp$, the space perpendicular to the kernel of $f$, is isomorphic to $\mathbb{V}/\ker f$, we can show they have the same dimension.

From our toy index theorem:

$0$ from above:

$\dim \ker \hat{f} + \dim \text{im} \hat{f} = \dim \mathbb{V}/\ker f$.

$\implies \dim \text{im} \hat{f} = \dim \text{im} f = \dim \mathbb{V}/\ker f$.

$\implies \dim \mathbb{V}/\ker f = \dim \text{im} f = \dim \mathbb{V} - \dim \ker f = \dim \mathbb{V} - \dim U$.

On the other hand, it's clear that

$\dim (U^\perp) = \dim \mathbb{V} - \dim U$.

So indeed,

$\dim \mathbb{V}/\ker f = \dim (U^\perp)$.

To specify the isomorphism, simply specify how to map basis vectors in one space onto those of the other.
2) a) Consider the map \( r : V^* \to U^* \) which is given by restriction, \( r(v) = v|_U \). Again our linear index theorem gives

\[
\dim \ker r + \dim \text{img } r = \dim V^* = \dim V
\]

by assumption

that \( \alpha \in X^* \) vanishes on \( U \).

Then,

\[
\dim X^* + \dim U = \dim V
\]

b) Well we know that the \( \ker f \) is a linear subspace of \( V \), what can we say about its annihilator space? Well the annihilator space should be a linear subspace of \( V^* \), so let's guess that it's \( \text{img } f^* \) and check if this is true. Let \( \ker f = U \) and \( X^* \) be \( X^* = \{ x \in V^* : f(x|_U) = 0 \} \).

Then for \( \beta \in \text{img } f^* \subset V^* \) we have \( \beta = f^*(\gamma) \) for some \( \gamma \in W^* \) and

\[
\beta(\gamma) = (f^*(\gamma))(\gamma) = f(f(\gamma)) = 0 \quad \forall \gamma \in U \quad \text{since } f(\gamma) = 0
\]

for these vectors, so \( \text{img } f^* \subset X^* \).

Now, suppose instead \( \beta \in X^* \) and define \( \gamma \in W^* \) by

\[
\gamma(\omega) = \beta(\omega) \quad \text{with } f(\omega) = 0.
\]

This definition is independent of \( \gamma \) because if we suppose \( f(\gamma) = f(\gamma') = 0 \)

then \( f(\gamma - \gamma') = 0 \) and \( \gamma - \gamma' \in \ker f \). But then

\[
\beta(\gamma') = \beta(\gamma + \gamma' - \gamma) = \beta(\gamma') \quad \text{because } \beta \in X^*.
\]

Finally we have \( \beta = f^* \gamma \) and so \( X^* \subset \text{img } f^* \).

Then \( \text{img } f^* = X^* \) and from part a) we have,

\[
\dim \ker f + \dim \text{img } f = \dim V
\]

\[
\dim \text{img } f = \dim V - \dim \ker f = \dim X^* + \dim \ker f - \dim \ker f = \dim \text{img } f^*
\]
3) We can just check directly that (4) satisfies (5):

\[ \langle \tilde{f} \omega, v \rangle_g = \langle (g^* f^* G) \omega, v \rangle_g \]

\[ = \langle (f^* G) \omega, v \rangle \text{ standard pairing} \]

\[ = \langle G(\omega), f(v) \rangle \text{ def. of } f^* \]

\[ = \langle \omega, f v \rangle_g \checkmark \]

4) a)

The cycle \( c \) is the boundary of the left or right halves but is not continuously deformable to a point.

b)

The cycles \( c_1 \) and \( c_2 \) are homologous as they bound the middle handle but cannot be deformed into one another.