Physics 250
Homework \#1

1) ai) Wed like to show that the relation $y=\Phi_{g} x$ is an equivalence relation, namely reflexive, symmetric and transitive. Every group is required to have cen identity element and the group action of this element is the identity mapping; then

$$
x=\Phi_{e} x \quad \Rightarrow \quad x \sim x
$$

and our relation is reflexive. If we have the relation $y=\Phi_{g} x$ then acting on both sides by $\Phi g^{-1}$ gives,

$$
\Phi_{g^{-1}} y=\Phi_{g^{-1}} \bar{\Phi}_{g} x=\Phi_{g^{-1} g} x=\Phi_{e} x=x
$$

and, because we are gourenteed the existance of $g^{-1}$, this gaurentees that

$$
x \sim y \quad \Leftrightarrow \quad y \sim x .
$$

Finally, if $y=\Phi_{g} x$ and $z=\Phi_{n} y$ then dy substitution

$$
z=\Phi_{k y}=\Phi_{k} \Phi_{y} x=\Phi_{k y} x=\Phi_{k} x
$$

aud because $k \in C$ is gamentened by closure this shows that

$$
x \sim y \text { and } y \sim z \Rightarrow x \sim z
$$

Thin's completes the proof of the fact that the relation is on equivalence relation.
b) Let's treat the case of the right coset. Introduce a group action of $H$ on $G$ given by multiplying elements of $G$ by elements of $H$ from the left,

$$
\Phi_{h}: G \rightarrow G \quad g \mapsto h g
$$

This is a group action as $\mathbb{Q}_{e}=$ identity on $G$ and $\Phi_{g} \Phi_{h} g=h^{\prime} h g=\Phi_{\text {hing }}$ and $h^{\prime} h \in H$. Now, the right coset of $g$ is the set

$$
[g]=\{\log \mid h \in H\}
$$

Another representative of the same coset $g^{\prime}$ must satisfy

$$
\left[g^{\prime}\right]=[g]
$$

or $g^{\prime}=h g$ for some $h \in H$
Then $g^{\prime}=\Phi_{h} g$ which shows that $g$ and $g^{\prime}$ lie on the same orbit of our grompaction. The converse is immediate if $g$ and $g^{\prime}$ lie on the same orbit then

$$
g^{\prime}=\Phi_{n} g=h g
$$

which shows that they are also in the same coset. The argument far lett cosets is similar, you can take the group action $\Phi_{n}: C_{4} \rightarrow C_{4}$

$$
\underline{\Phi}_{h g}=g h^{-1} \text {. }
$$

You mate this choice to ensure that In is a left action.

1) c) Nakcehara 2.6: For technical reasons explained below it is nicer to consider $\binom{\tau}{1} \in \mathbb{C} P^{\prime}$, recall that $\mathbb{C} p^{\prime}$ consists of elements $\binom{a}{b} \in \mathbb{C}^{2}$ that are identified under the equivalence retation $\binom{a}{b} \sim\binom{\lambda a}{\lambda b} \lambda \in \mathbb{C}$, so that in particular $\binom{a}{b} \sim\binom{a / b}{1}$.

Now, take the group action $\Phi: A \mapsto \Phi_{A}$ with $A \in S L(2, \mathbb{Z})$,

$$
\binom{\tau^{\prime}}{1}=\Phi_{A}\binom{\tau}{1}=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)\binom{\tau}{1}=\binom{a \tau+b}{c \tau+d} \sim\binom{\frac{a \tau+b}{c \tau+d}}{1}
$$

for

First note that,

$$
\begin{gathered}
A=\left(\begin{array}{ll}
a & b \\
c
\end{array}\right) \\
\tau \in H
\end{gathered}
$$

$$
\begin{aligned}
2 \operatorname{Im} \tau^{\prime} & =\tau^{\prime}-\tau^{\prime *}=\frac{a \tau+b}{c \tau+d}-\frac{a \tau^{*}+b}{c \tau^{*}+d}=\frac{(a d-b c)\left(\tau-\tau^{*}\right)}{|c \tau+d|^{2}} \\
& =\frac{2 \operatorname{Im}(\tau)}{\mid c \tau+d\left(^{2}\right.}>0
\end{aligned}
$$

so indeed $\tau^{\prime} \in H H$. Now, check that $\Phi_{A}$ is a group action, $\bar{\phi}_{i d}\binom{\tau}{1}=\binom{1 \cdot \tau+0 / 0 \cdot+1}{1}=\binom{\tau}{1}$ as needen.
and given $A=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right), \quad B=\left(\begin{array}{ll}a & f \\ g & k\end{array}\right) \in S L(z, \mathbb{Z})$

$$
\Phi_{A} \Phi_{B}\binom{\tau}{1}=A B\binom{\tau}{1}=\Phi_{A B}\binom{\tau}{1}
$$

Then $\Phi_{A}$ is a lett group action on $H$. and indeed $\tau^{\prime} \sim \tau$ are on the same group orbit. Finally, the meson for introducing the projective coordinates is that even cohen $c \tau+d=0$ the group action is well defined and takes $\binom{\tau}{1} \mapsto\binom{a \tau+b}{0}$ which is a print at $\infty$ in the projective space. This group action is important var for clliptic curves hence modular forms hence the proof of Fermat's last theorem.
 this five the closure is frivial and we only check the

$$
\Phi_{c} g=e g e^{-1}=g \alpha \text { and } \Phi_{g} \Phi_{h} g^{\prime}=g^{h} g^{\prime} h^{-1} g^{-1}=\Phi_{g h} g^{\prime}
$$

So $\Phi$ is a group action and its orbits are the equivalence classes generated by conjugation.
2) If $\operatorname{SO}(2) \subset S O(3)$ is the sorbroup of rotations about the $z$-axis then an element of the left coset will have the form,

$$
\begin{aligned}
{[R] } & =R R_{z}(\lambda) \quad R_{z}(\lambda) \in S O(2) \\
& =R_{z}(\alpha) R_{y}(\beta) R_{z}(\gamma) R_{z}(\lambda) \\
& =R_{z}(\alpha) R_{y}(\beta) R_{z}(\gamma+\lambda)
\end{aligned}
$$

Hence, for two elements of the left coset $R_{1}$ and Re we have

$$
\alpha_{1}=\gamma_{2} \text { } \beta_{1}=\beta_{2} \text { and } \gamma_{2}=\gamma_{1}+\lambda \text { for some }
$$

$$
\lambda \in(0,2 \pi)
$$

Similarly, for the right coset we have,
$\beta_{1}=\beta_{2} \quad \gamma_{1}=\gamma_{2}$ and $\gamma_{2}=\alpha_{1}+\rho$ for some

$$
\rho \in[0,2 \pi) \text {. }
$$

The topology of the quotient space $\delta 0(3) / \delta O(2)$ is the same as that of $S^{2}$. This can be exhibited by considering the map
$H: S O(3) / S_{(2)} \rightarrow s^{2} \quad[\&] \rightarrow \mathbb{z}$.
This map is onto as any point on the sphere can be reached by a rotation $R$. The map is also 1 -toll for if $R \hat{z}=R^{\prime} \hat{z} \Rightarrow \hat{z}=R^{-1} R^{\prime} \hat{z}$, so $R^{-1} R^{\prime} \in S O(2)$ which shows that $[R]=\left[R^{\prime}\right]$. If we claim without proof that is is continuous (it is) this shans the te $H$ is a homeomorphism and so $50(3) / S 0(2)$ has the same topology as $S^{2}$.
The intuition here is that if you start with any vector $\vec{x} \in \mathbb{R}^{x}$ and make it equivalent to all vectors that differ by $R_{z}$ rotation you've forgotten its $x$ and $y$ components.

# Hopf Fibration and the 2D Oscillator 

Hal Haggard

## 1 Visualizing the three sphere

Nota Bene: This is a write-up from my younger more naive days. I hope you all enjoy it! HMH
In the physics literature the Hopf bundle is most often discussed in connection with Dirac's argument that the existence of a magnetic monopole would explain the quantization of electric charge. Here I would like to give an elementary geometric description of the Hopf map and discuss its connection to the 2D harmonic oscillator. This is an amalgamation of many results present in the literature.

Hopf's procedure shows how to project the three sphere $S^{3}$, which consists of all points of $\mathbb{R}^{4}$ with unit distance from the origin, onto the two sphere $S^{2}$, the usual surface of a ball in space, $\mathbb{R}^{3}$. The inverse image of a projected point $p \in S^{2}$ is an entire circle of points $S^{1} \subset S^{3}$. The full argument shows that $S^{3}$ is a circle or $U(1)$ bundle over the base space $S^{2}$; the circles sitting over each point of the $S^{2}$ are linked one with the next, once and only once. There is also a sense in which a set of tori that appear in the description of the the three sphere are linked.

Let us begin by visualizing $S^{3}$, represent a point of $\mathbb{R}^{4}$ by the cartesian vector $\left(x_{1}, p_{1}, x_{2}, p_{2}\right)$. Notation simplifies a bit if we consider this vector as a pair of complex numbers $z_{1} \equiv x_{1}+i p_{1}$, $z_{2} \equiv x_{2}+i p_{2}$, so that the vector $\left(z_{1}, z_{2}\right)$ is an element of $\mathbb{C}^{2} \cong \mathbb{R}^{4}$. For this vector to lie on the three sphere it must satisfy,

$$
\left|z_{1}\right|^{2}+\left|z_{2}\right|^{2}=1
$$

or in terms of a polar representation $z_{1}=r_{1} e^{i \xi_{1}}, z_{2}=r_{2} e^{i \xi_{2}}$,

$$
r_{1}^{2}+r_{2}^{2}=1
$$

This relation is conveniently parametrized in terms of trigonometric functions, let $r_{1}=\cos (\theta / 2)$ and $r_{2}=\sin (\theta / 2)$, where the range of $\theta$ is $\theta \in[0, \pi]$ in order that $r_{1}$ and $r_{2}$ are always positive. Now a point of $S^{3}$ is given by

$$
\left(\cos \left(\frac{\theta}{2}\right) e^{i \xi_{1}}, \sin \left(\frac{\theta}{2}\right) e^{i \xi_{2}}\right) \quad \theta \in[0, \pi], \quad \xi_{1}, \xi_{2} \in \mathbb{R}
$$

For fixed $\theta$, this is the cartesian product of two circles and so generically describes a torus. A nice way to visualize these tori is by stereographic projection $P: S^{3} \subset \mathbb{R}^{4} \rightarrow \mathbb{R}^{3} \cup\{\infty\}$. If we project from the south pole $(0,0,0,-1)$ then we have for $q \in S^{3}$,

$$
\begin{aligned}
P(q) & =\left(\frac{x_{1}}{1+p_{2}}, \frac{p_{1}}{1+p_{2}}, \frac{x_{2}}{1+p_{2}}\right) \\
& =\left(\frac{\cos \left(\frac{\theta}{2}\right) \cos \left(\xi_{1}\right)}{1+\sin \left(\frac{\theta}{2}\right) \sin \left(\xi_{2}\right)}, \frac{\cos \left(\frac{\theta}{2}\right) \sin \left(\xi_{1}\right)}{1+\sin \left(\frac{\theta}{2}\right) \sin \left(\xi_{2}\right)}, \frac{\sin \left(\frac{\theta}{2}\right) \cos \left(\xi_{2}\right)}{1+\sin \left(\frac{\theta}{2}\right) \sin \left(\xi_{2}\right)}\right)
\end{aligned}
$$



Figure 1: A cutaway side view of several tori in the range $0<\theta \leq \pi / 2$. The white circle represents the degenerate torus as $\theta$ goes to 0 . The yellow z-axis represents the degenerate torus as $\theta$ goes to $\pi$.

As we will see below, the value $\theta=\pi / 2$ marks a boiundary between two qualitatively different sets of tori. As the parameter $\theta$ ranges from zero to $\pi / 2$ it sweeps out the nested toric leaves of a solid torus $D \times S^{1}$, see Figure 1. The parameter value $\theta=0$ is a circle, viewed in this context as a degenerate torus. The value $\theta=\frac{\pi}{2}$ gives a boundary torus for which $r_{1}=r_{2}=\frac{\sqrt{2}}{2}$, we'll call this boundary torus $T$. The parameter values, $\theta \in\left[\frac{\pi}{2}, \pi\right]$ also sweep out a solid torus. However these tori degenerate into a circle on $S^{3}$ that goes through the south pole and hence is projected onto an entire line, in our coordinates the z-axis. The two degenerate circles are also depicted in Figure 1. Because of this projective oddity viewing these tori as nested requires a dual point of view; you look at the tori from "inside", see Figure 2. This can also be visualized by choosing a different


Figure 2: The solid torus viewed from "inside". An outer torus has been cut up so that you can see the torus lying beneath it. As you proceed further in to deeper tori eventually you reach the degenerate line depicted in yellow in Figure 1.
point from which to stereographically project, see Figure 3. The Tori are also clearly linked with one another from this point of view. This completes our visualization of $S^{3}$, we proceed to the Hopf fibration.


Figure 3: Two tori, one from each family are projected after a $\pi / 2$ rotation in the $p_{1} p_{2}$ plane. The transition from one family to the other is shown as a cut away torus transitions through various values of $\theta$. The central picture is the bounding torus $T$.

## 2 Hopf fibration

There is a very simple group action of the circle, denoted in this section by $U(1)$, on the three sphere. This action simply multiplies all the coordinates by an element $g \in U(1)$,

$$
g \cdot q=g \cdot\left(z_{1}, z_{2}\right)=\left(g \cdot z_{1}, g \cdot z_{2}\right)
$$

This is a group action, which can be demonstrated simply:

$$
g_{1} \cdot\left(g_{2} \cdot q\right)=\left(g_{1} g_{2}\right) \cdot q \quad e \cdot q=q .
$$

The orbit of a point $q$ is given by $q \cdot g$ as $g$ takes on each value in $U(1)$. The idea of the Hopf map is to identify all of the points on this orbit - to quotient out the $S^{1}$ generated by the group action described above. The group action becomes trivial when acting on ratios of the two complex coordinates. Let us define the map $r: \mathbb{C}^{2} \rightarrow \mathbb{C}^{*}$ by

$$
r\left(z_{1}, z_{2}\right)=\frac{z_{1}}{z_{2}}=\frac{r_{1}}{r_{2}} e^{i\left(\xi_{1}-\xi_{2}\right)} .
$$

This map takes us down from the three sphere to the projective plane. To complete the definition of the Hopf map we use the inverse of stereographic projection to take the points of the plane to the sphere. The inverse projection from the north pole is given by,

$$
\begin{aligned}
\phi_{N P}^{-1}(x, y) & =\left(\frac{2 x}{1+x^{2}+y^{2}}, \frac{2 y}{1+x^{2}+y^{2}}, \frac{x^{2}+y^{2}-1}{1+x^{2}+y^{2}}\right) \\
& =\left(\frac{z+\bar{z}}{z \bar{z}+1}, \frac{i(\bar{z}-z)}{z \bar{z}+1}, \frac{z \bar{z}-1}{z \bar{z}+1}\right)
\end{aligned}
$$

The composition of these two maps gives us the Hopf map, $h=\phi_{N P}^{-1} \circ r: S^{3} \rightarrow S^{2}$,

$$
h\left(z_{1}, z_{2}\right)=\left(z_{1} \bar{z}_{2}+\bar{z}_{1} z_{2}, i\left(\bar{z}_{1} z_{2}-z_{1} \bar{z}_{2}\right),\left|z_{1}\right|^{2}-\left|z_{2}\right|^{2}\right) .
$$

We pause briefly to compute this in our coordinates on $\mathbb{R}^{4}$ and in the parametric coordinates $\left(\theta, \xi_{1}, \xi_{2}\right)$ :

$$
h\left(x_{1}, p_{1}, x_{2}, p_{2}\right)=\left(2\left(x_{1} x_{2}+p_{1} p_{2}\right), 2\left(x_{2} p_{1}-x_{1} p_{2}\right), x_{1}^{2}+p_{1}^{2}-x_{2}^{2}-p_{2}^{2}\right) .
$$

In the parametric coordinates we have,

$$
\begin{aligned}
z_{1} \bar{z}_{2}+\bar{z}_{1} z_{2} & =\cos \left(\frac{\theta}{2}\right) \sin \left(\frac{\theta}{2}\right)\left[e^{i\left(\xi_{1}-\xi_{2}\right)}+e^{-i\left(\xi_{1}-\xi_{2}\right)}\right] \\
& =\frac{1}{2} \sin \theta\left[2 \cos \left(\xi_{1}-\xi_{2}\right)\right]=\sin \theta \cos \phi,
\end{aligned}
$$

where the shorthand $\phi \equiv \xi_{1}-\xi_{2}$ has been introduced. The others are,

$$
\begin{aligned}
h\left(\theta, \xi_{1}, \xi_{2}\right) & =\left(\sin \theta \cos \phi, \cos \left(\frac{\theta}{2}\right) \sin \left(\frac{\theta}{2}\right) i\left[e^{-i\left(\xi_{1}-\xi_{2}\right)}-e^{i\left(\xi_{1}-\xi_{2}\right)}\right], \cos ^{2}\left(\frac{\theta}{2}\right)-\sin ^{2}\left(\frac{\theta}{2}\right)\right) \\
& =(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)
\end{aligned}
$$

which is immediately recognizable as the polar parametrization of the sphere $S^{2}$.

### 2.1 2D Oscillator

The Lagrangian for the 2 D harmonic oscillator is,

$$
L=\frac{1}{2} m\left(\dot{x}_{1}^{2}+\dot{x}_{2}^{2}\right)-\frac{1}{2} k\left(x_{1}^{2}+x_{2}^{2}\right) .
$$

Generically the spring constant and masses of the oscillators are different. Here we'll consider the case where they are the same, this is sometimes called the 1:1 resonance case. We can choose units of time such that the natural frequency $\omega_{0}=\sqrt{k / m}$ is equal to one and units of mass such that $m=1$; then we also have $k=1$. In this system of units the Lagrangian is

$$
L=\frac{1}{2}\left(\dot{x}_{1}^{2}+\dot{x}_{2}^{2}-x_{1}^{2}-x_{2}^{2}\right) .
$$

The canonical momenta are,

$$
p_{1}=\frac{\partial L}{\partial \dot{x}_{1}}=\dot{x}_{1} \quad p_{2}=\frac{\partial L}{\partial \dot{x}_{2}}=\dot{x}_{2}
$$

and the Hamiltonian is,

$$
H=p_{i} \dot{x}_{i}-L=\frac{1}{2}\left(x_{1}^{2}+p_{1}^{2}+x_{2}^{2}+p_{2}^{2}\right) .
$$

Hamilton's equations are,

$$
\begin{array}{rlrl}
\dot{x}_{1} & =\frac{\partial H}{\partial p_{1}}=p_{1} & \dot{p}_{1}=-\frac{\partial H}{\partial x_{1}}=-x_{1} \\
\dot{x}_{2}=\frac{\partial H}{\partial p_{2}}=p_{2} & \dot{p}_{2}=-\frac{\partial H}{\partial x_{2}}=-x_{2}
\end{array}
$$

The first set of these equations has the usual solution:

$$
\begin{array}{r}
x_{1}(t)=x_{1}(0) \cos t+p_{1}(0) \sin t \\
x_{1}(t)=p_{1}(0) \cos t-x_{1}(0) \sin t .
\end{array}
$$

This is a circle in the $x_{1}-p_{1}$ plane which is conveniently regarded as the complex plane $z_{1}=x_{1}+i p_{1}$ (Note: Our definitions lead to the slight oddity that the circle is traversed in a clockwise sense.)

Because the Lagrangian has no explicit time dependence the total energy $E=H$ is conserved. Each of the oscillator's energies is also separately conserved, so we can construct a second independent constant of the motion by considering the energy difference:

$$
J_{3}=\frac{1}{2}\left(x_{1}^{2}+p_{1}^{2}\right)-\frac{1}{2}\left(x_{2}^{2}+p_{2}^{2}\right) .
$$

Further, the angular momentum is conserved by the dynamics as is the quantity $2\left(x_{1} x_{2}+p_{1} p_{2}\right)$ which you can confirm by direct calculation. I called this last quantity the correlation in my undergraduate thesis. But these three quantities are precisely the coordinates on the quotient space calculated above. The $S^{2}$ of the hopf fibration is the space of motions of the 2 D harmonic oscillator.

## 3 Linkage of the Hopf fibers

I never got around to completing this section but there are some pretty pictures I made. The basic idea is that each of the fibers of the hopf fibration links all of the others once.


Figure 4: Broadened Hopf fibers are sequentially added to demonstrate how each fiber is linked to the next without crossing it.


Figure 5: The same fibered torus from a top view and a side view.

