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We see that a connection implies the existence of an Ehresmann form with properties (1)-(3) above. Note that  $H_u P = \ker(\alpha|_u)$ . Conversely, suppose we are given a  $\mathfrak{g}$ -valued 1-form with properties (2) and (3). We then define  $H_u P = \ker(\omega|_u)$ . It then follows that  $T_u P = V_u P \oplus H_u P$  and  $R_{\alpha^*}(H_u P) = H_{\alpha(u)} P$ . (Exercise for you). Thus we have three equivalent ways of specifying a connection:

- (1) Connection coefficients,  $\Gamma_{\alpha\beta}^{\gamma}$ , or  $\nabla$  operator on  $M$
- (2)  $\mathfrak{g}$ -invariant horizontal subspaces in  $TPB$
- (3) Ehresmann form  $\omega$ .

Now we discuss a fourth equivalent way, which uses gauge potentials. A gauge potential  $A$  is a  $\mathfrak{g}$ -valued 1-form defined on  $M$ , not  $P$  (as  $\omega$ ). Actually, it is only defined over a local chart  $U \subset M$ , and the definition is made relative to a local section (in  $P$ ) over  $U$ . It is thus intrinsically a coordinate-dependent object (i.e., section-dependent, but putting a local section in  $P$  is tantamount to choosing coordinates on  $P$ ). For this reason we have to study how  $A$  transforms when we change local section (this is called a gauge transformation). Also, introducing a section complicates the geometry, as we will see. For all these reasons, purists (= mathematicians) prefer to work with  $\omega$ , which has an intrinsic (geometrical) meaning on  $P$ , independent of coordinates. Physicists (as they say) prefer to work with gauge potentials, because often  $M$  is regarded as the "real" space

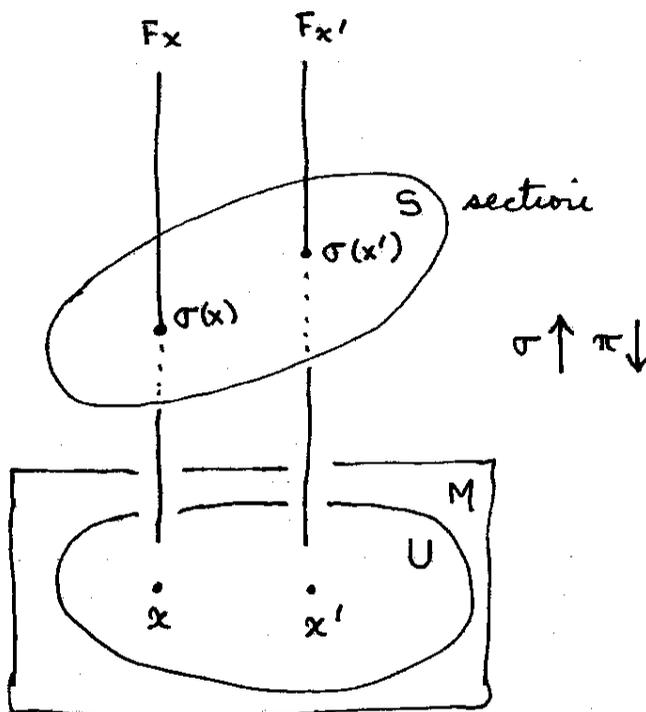
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we live on (i.e. space-time), where the potentials  $A$  are defined. (But one can question whether the "real" space is actually a PFB over  $M$ , where  $\omega$  lives.) Also, vector potentials have a long tradition in physics.

A local section of a PFB is a map  $\sigma: U \rightarrow P$ , such that  $\pi \sigma(x) = x$ . A picture:

The surface  $S$  is the image of the map  $\sigma$  (we call also  $S$  the "section").  $U$  and  $S$  are diffeomorphic, in fact the projection  $\pi: P \rightarrow M$ , restricted to  $S$ , is the inverse of  $\sigma$ . The section  $S$  need only be a smooth



surface in  $P$  above  $U$ , which intersects each fiber transversally at one point. Thus there is a huge infinity of possible ways to choose a section, and anything dependent on it (like  $A$  or the coordinate  $g$  to be introduced momentarily) has a large degree of arbitrariness in it.

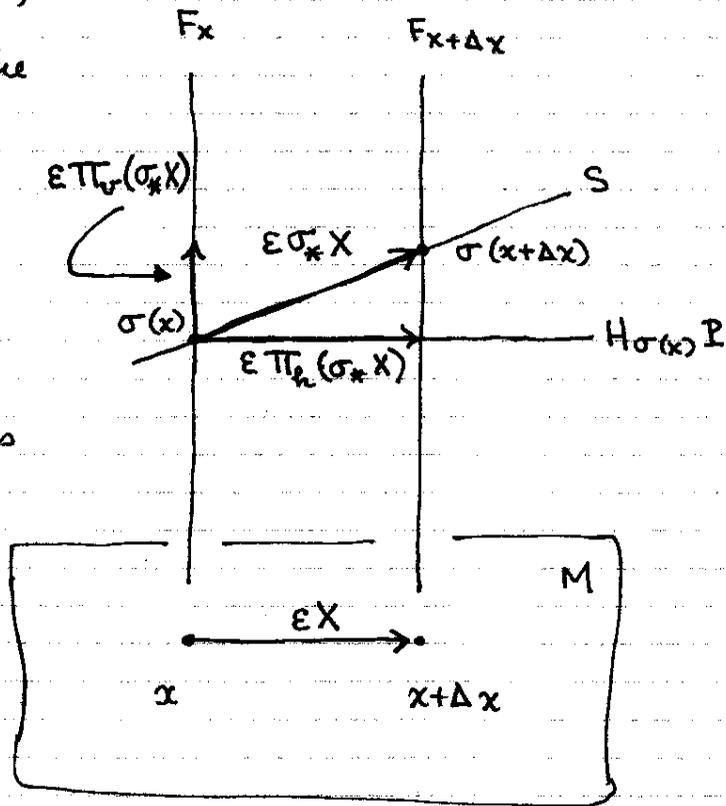
Supposing now that an Ehresmann form is given on  $P$ , we define

$$A = \sigma^* \omega$$

which makes  $A$  a field of maps,  $A|_x: T_x M \rightarrow \mathfrak{g}$ , for  $x \in U$ , i.e.,  $A \in \mathfrak{g} \otimes \Omega^1(U)$ .

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To interpret  $A$ , make a more schematic drawing of two nearby fibers in  $P$  and the section  $S$ . Look at fibers over  $x$  and  $x+\Delta x$  in  $M$ . Write  $\Delta x = \epsilon X$ , where  $X \in T_x M$ .  $\sigma(x)$  is the point on the section over  $x$ , and the horizontal subspace  $H_{\sigma(x)} P$  is drawn in. It is drawn at right angles on the paper, but this has no significance (in general there is no metric on  $P$ , even if there is one on  $M$ ).



The section is drawn at an angle to the horizontal subspace, because  $S$  is highly arbitrary and there is no reason why it (i.e., its tangent plane) should be horizontal.

By the definition of  $A$  and the pull-back, we have

$$A|_x(X) = (\sigma^*\omega)|_x(X) = \omega|_{\sigma(x)}(\sigma_*X).$$

But  $\epsilon\sigma_*X$  is the vector in  $T_{\sigma(x)}P$  that is tangent to the section  $S$  and connects the same fibers as  $\epsilon X$ . Thus  $\omega|_{\sigma(x)}$  acting on this projects out the vertical part and converts it to an element of  $\mathfrak{g}$ . In the diagram,

$$\pi_v(\sigma_*X) = A|_x(X) \#$$

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Thus we can say that  $A|_x(X)$  is a measure of the "slope" of the section  $S$  in the bundle at the point over  $x$  in the direction  $X$ , that is,  $A|_x$  measures how much the section deviates from the horizontal at  $x$ . In particular,  $A|_x = 0$  iff the section (i.e., its tangent plane) is purely horizontal at  $x$ .

We can always choose a section  $S$  to be purely horizontal over one point  $x \in M$ , but we cannot choose it to be horizontal over a region in  $M$  unless the distribution specified by  $H\pi$  is integrable. In that case, a purely horizontal section can be chosen, and  $A = 0$  over the region in question.

The definition of  $A$  gives  $A$  in terms of  $\omega$ . Can we find an expression giving  $\omega$  in terms of  $A$ ? The answer is yes, because by the property  $R_a^* \omega = A da^{-1} \omega$ , to specify  $\omega$  ~~at a point~~ on a fiber it suffices to know  $\omega$  (that is, to know its action on arbitrary tangent vectors) at any one point of the fiber. If  $A$  is given, the simplest point to choose is (on  $F_x$ ) is  $\sigma(x)$ .

In fact, by the formula above, the action of  $\omega|_{\sigma(x)}$  on any vector tangent to  $S$  can be computed in terms of  $A|_x$ . Also, by the definition of  $\omega$ , its action on vertical vectors is known. So, we need to take an arbitrary vector  $Y \in T_{\sigma(x)} P$ , break it into vertical and section (= tangent to section) components, and then  $\omega|_{\sigma(x)}(Y)$  is computable. This is a different decomposition than that considered earlier (the horizontal-vertical) decomposition. In particular, when we write

$$Y = Y_S + Y_V$$

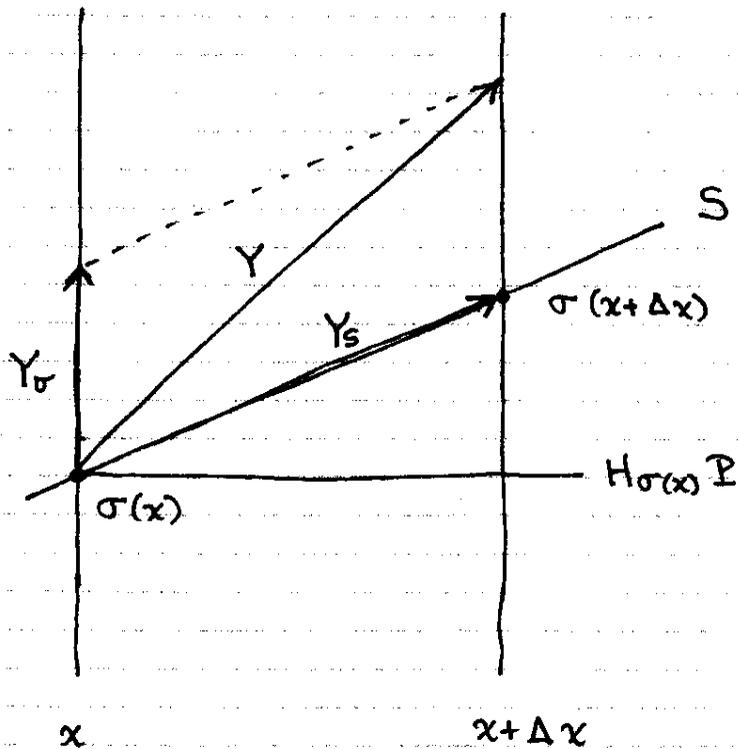
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for the section and vertical components of  $Y$ , the vertical part  $Y_v$  is not the same as the  $Y_v$  we had earlier, when we wrote  $Y = Y_h + Y_v$ . This is clear from a picture: (You can see that  $Y_v$  is not the same as we would have in an  $h-v$  decomposition.)

So, we can write

$$\omega|_{\sigma(x)}(Y) = \omega|_{\sigma(x)}(Y_s) + \omega|_{\sigma(x)}(Y_v).$$



Of these, the first (s) term is easy, since if we write

$$X = \pi_* Y,$$

$$X \text{ also } = \pi_* Y_s, \text{ but } Y_s = \sigma_* X, \text{ so}$$

$$Y_s = \sigma_* \pi_* Y,$$

and

$$\omega|_{\sigma(x)}(Y_s) = \omega|_{\sigma(x)}(\sigma_* \pi_* Y)$$

$$= (\sigma^* \omega)|_x(\pi_* Y)$$

$$= A|_x(\pi_* Y).$$

For this term (the s-term), we just project  $Y$  onto  $M$  and use  $A$  on it.

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For the other (v) term it helps to introduce explicit coordinates on the bundle  $P$ . If  $u \in P$  (over  $U \subset M$ ) we will write the coordinates of  $u$  as  $(x, a)$ , where  $x \in M$  and  $a \in G$ , by writing

$$u = \sigma(x)a, \quad x = \pi(u)$$

Thus  $a$  is the group element needed to reach  $u$  from the section by the right action of  $G$  on  $P$ :

Obviously the  $a$ -coordinate of  $u$  depends on the section, while the  $x$ -coordinate does not. Let us write

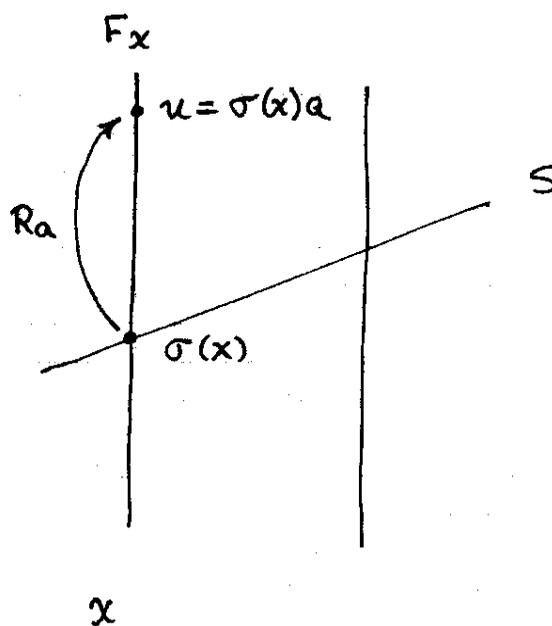
$$\begin{aligned} g: \pi^{-1}(U) &\rightarrow G \\ &: u \mapsto a \end{aligned}$$

for the coordinate function (distinguishing the function  $g$  from its value  $a$ ). This

is just another notation for the local trivialization associated with the section  $S$ ,

$$\begin{aligned} \Phi: U \times G &\rightarrow \pi^{-1}(U) \\ &: (x, a) \mapsto \sigma(x)a \end{aligned}$$

$$\begin{aligned} \text{or } \phi_x: G &\rightarrow F_x \\ &: a \mapsto \sigma(x)a \end{aligned}$$



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what Nak. calls the "canonical local triad", so that

$$g(u) = \phi_{i,x}^{-1}(u), \quad \text{where } x = \pi(u),$$

$$\text{or } g(\sigma(x)a) = a.$$

Notice that the section  $S$  is given by  $g(u) = e = \text{const.}$ , i.e.,

$$g(\sigma(x)) = g(\sigma(x)e) = e.$$

Consider now the tangent map  $g_*$ . If we evaluate this at  $u = \sigma(x)a$ , we have

$$g_*|_u : T_u P \rightarrow T_a G.$$

In particular, setting  $u = \sigma(x)$ ,  $a = e$ , we get

$$g_*|_{\sigma(x)} : T_{\sigma(x)} P \rightarrow \mathfrak{g}.$$

Evaluated at  $\sigma(x)$ ,  $g_*$  is a map very much like  $\omega|_{\sigma(x)}$ , i.e., it is a Lie-algebra-valued form. However, the kernel of  $g_*$  is the tangent space to the section  $T_{\sigma(x)} S$  (the space of  $S$ -vectors), not  $H_{\sigma(x)} P$  as is the case with  $\omega|_{\sigma(x)}$ . This is because  $g$  is constant on  $S$ , so  $g_*(Y_S) = 0$  for any  $Y_S$  tangent to  $S$ .

As for vertical vectors, we have

$$g_*|_{\sigma(x)} (V^\#|_{\sigma(x)}) = V,$$

something which is almost obvious with a little intuition (think: how much does  $g$  vary when we move a small distance in a vertical direction away from  $\sigma(x)$ ). But a formal proof is easily given, using equivalence classes of curves for vectors:

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$$\begin{aligned}
 g_*|_{\sigma(x)} (V^\#|_{\sigma(x)}) &= g_*|_{\sigma(x)} [\sigma(x) \exp(tV)] \\
 &= [g(\sigma(x) \exp(tV))] \\
 &= [\exp(tV)] = V.
 \end{aligned}$$

$g_*|_{\sigma(x)}$  has the same action on vertical vectors as  $\omega|_{\sigma(x)}$ ,

$$\omega|_{\sigma(x)} (Y_V) = g_*|_{\sigma(x)} (Y_V).$$

But, since  $g_*|_{\sigma(x)} (Y_S) = 0$ , the above is also  $g_*|_{\sigma(x)} (Y)$ .

Thus in our s-v decomposition, we have

$$Y = Y_S + Y_V \in T_{\sigma(x)}\mathbb{P}$$

$$\begin{aligned}
 \omega|_{\sigma(x)}(Y) &= \omega|_{\sigma(x)}(Y_S) + \omega|_{\sigma(x)}(Y_V) \\
 &= A|_x(\pi_* Y) + g_*|_{\sigma(x)}(Y) \\
 &= (\pi^* A)|_{\sigma(x)}(Y) + g_*|_{\sigma(x)}(Y).
 \end{aligned}$$

This specifies the action of  $\omega|_{\sigma(x)}$  on an arbitrary vector  $Y \in T_{\sigma(x)}\mathbb{P}$ .

Now consider the action of  $\omega$  at an arbitrary point  $u = \sigma(x)a$ . Let  $Z \in T_u\mathbb{P}$  be an arbitrary tangent vector at  $u$ . By the properties of  $\omega$  we have

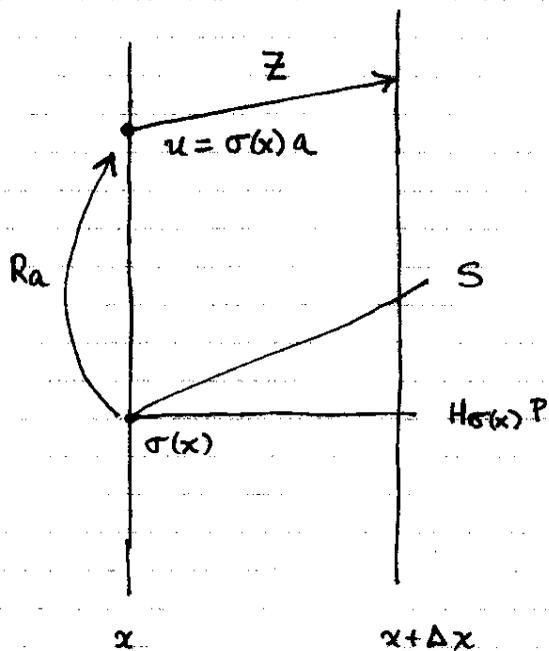
$$\begin{aligned}
 \omega|_u(Z) &= \omega|_{\sigma(x)a}(Z) = (R_a^* \omega)|_{\sigma(x)}(R_a^{-1*} Z) \\
 &= Ad_{a^{-1}} \omega|_{\sigma(x)}(R_a^{-1*} Z).
 \end{aligned}$$

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So, we use the formula above for  $\omega|_{\sigma(x)}$  and set

$$Y = R_a^{-1} * Z.$$

This gives,



$$\omega|_u(Z) = Ad_{a^{-1}} \left[ A|_x (\pi_* R_a^{-1} * Z) + g_* R_a^{-1} * Z \right].$$

This can be simplified. First note that

$$\pi R_a^{-1} = \pi,$$

so

$$\pi_* R_a^{-1} * = \pi_*^*$$

and the  $R_a^{-1} *$  can be dropped in the first term. Next, write an arbitrary element of  $P$  as  $\sigma(x)b$  for some  $b \in G$ , and consider

$$\begin{aligned} g R_a^{-1} (\sigma(x)b) &= g (\sigma(x)ba^{-1}) \\ &= ba^{-1} \\ &= g(\sigma(x)b)a^{-1} \\ &= R_a^{-1} g (\sigma(x)b), \end{aligned}$$

i.e.,  $g$  and  $R_a^{-1}$  commute, so

$$g_* R_a^{-1} * = R_a^{-1} * g_*.$$

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Finally, note that  $Ad_{a^{-1}} = L_{a^{-1}} \circ R_{a^*}$ . Altogether, this gives

$$\omega|_u(Z) = Ad_{a^{-1}} \left( A|_x(\pi^*Z) \right) + L_{a^{-1}} \circ g_* Z,$$

or

$$\omega|_u = Ad_{a^{-1}}(\pi^*A)|_u + L_{a^{-1}} \circ g_*|_u,$$

where it is understood that  $a = g(u)$ ,  $u = \sigma(x)a$ . This is the explicit expression for  $\omega$  in terms of  $A$ .

Nak. writes it this way:

$$\omega = g^{-1}(\pi^*A)g + g^{-1}dg.$$

He is not distinguishing between the function  $g$  and the value  $a$ , and he is thinking of a matrix group where  $Ad_{a^{-1}}b = a^{-1}ba$  and where  $L_{a^{-1}}$  is just mult. by  $a^{-1} = g^{-1}$ . Also, we have never defined  $dg$  for a map like  $g: \pi^{-1}(U) \rightarrow G$ , but if you think of  $g$  as a " $G$ -valued 0-form" then  $dg$  means the same as  $g_*$  (or, ~~the~~ thinking of  $g$  as a matrix-valued function, then  $dg$  is a matrix of 1-forms).

Now it is possible to show that if  $A$  is a given  $\mathfrak{g}$ -valued 1-form on  $U$ , and if the section  $\sigma$  and associated function  $g$  are given, and if we define  $\omega$ , a  $\mathfrak{g}$ -valued 1-form on  $\pi^{-1}(U)$  by the boxed formula above, then  $\omega$  satisfies the two requirements of an Ehresmann form,

$$\omega|_u(V^*|_u) = V,$$

$$R_b^* \omega = Ad_{b^{-1}} \omega,$$

(Exercise for you).

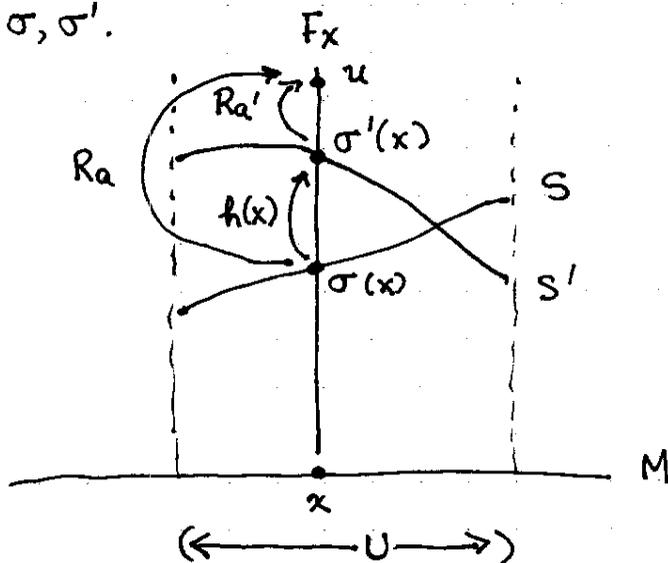
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So if we are given an Ehresmann form  $\omega$  and we define  $A$  relative to a local section by  $A = \sigma^* \omega$ , then the boxed formula on the previous page expresses  $\omega$  at an arbitrary point  $z \in \pi^{-1}(U)$  acting on an arbitrary vector  $\in T_z P$  in terms of  $A, \sigma$ , and the associated coordinate function  $g$ .

Alternatively, if  $A, \sigma, g$  are given and we use the formula in the box to define  $\omega$ , then  $\omega$  satisfies conditions at the bottom of the previous page, which any Ehresmann connection must satisfy. Does this mean that  $\omega$  defined this way is an Ehresmann connection? No, because the latter is defined everywhere on  $P$  and  $\omega$  defined in the boxed eqn is only defined over  $\pi^{-1}(U)$ .

So we need to cover  $M$  with open sets  $\{U_i\}$ , define local sections  $\sigma_i: U_i \rightarrow P$  over each and coordinate fuz.  $g_i: \pi^{-1}(U_i) \rightarrow G: \sigma_i(x)a \mapsto a$ , and then we need to check whether the  $\omega$ 's defined over the sets  $\pi^{-1}(U_i)$  agree in the overlap regions.

To deal with a slightly simpler problem, let's see how the gauge potential ~~changes when we change local sections~~  $A$  defined over a single region  $U$  changes when we change local sections. So we want one  $U$  but 2 local sections  $\sigma, \sigma'$ .



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Let  ~~$\sigma$~~   $\sigma: U \rightarrow P$  } be 2 local sections,  $S = \text{ring}(\sigma)$   
 $\sigma': U \rightarrow P$  }  $S' = \text{ring}(\sigma')$

with 2 associated coordinate functions,

$$g: \pi^{-1}(U) \rightarrow G: \sigma(x)a \mapsto a$$

$$g': \pi^{-1}(U) \rightarrow G: \sigma'(x)a \mapsto a$$

In the figure,  $g(u) = a$  and  $g'(u) = a'$ . Also,  $h(x)$  is the group element connecting the 2 sections at  $x$ , i.e.  $h: U \rightarrow G$ , and  $\sigma'(x) = \sigma(x)h(x)$ . From the picture,

$$u = R_a \sigma(x) = R_{a'} R_{h(x)} \sigma(x)$$

$$\Rightarrow R_a = R_{a'} R_{h(x)} = R_{h(x)} a'$$

$$\Rightarrow a = h(x) a' \quad \Rightarrow \quad g(u) = h(x) g'(u) \quad \text{where } x = \pi(u).$$

So this is the coordinate transformation.

Now in the boxed eqn, we have

$$\text{Ad}_{a^{-1}} = \text{Ad}_{a'^{-1}} h(x)^{-1} = \text{Ad}_{a'^{-1}} \text{Ad}_{h(x)^{-1}}$$

since  $\text{Ad}$  is a representation of  $G$ . So,

$$\omega|_u = \text{Ad}_{a'^{-1}} \left( \text{Ad}_{h(x)^{-1}} (\pi^* A)|_u \right) + \text{2nd term.}$$

For the second term, we have

$$L_{a^{-1}} = L_{a'^{-1}} L_{h(x)^{-1}}$$

As for  $g_*$ , since  $g(u) = h(x) g'(u)$ , you might think  
 $= L_{h(x)} g'(u)$ ,

$$g_* = L_{h(x)} g'_*$$

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by rules of tangent maps. But this misses the fact that  $x$  depends on  $u$  through  $x = \pi(u)$ , so when we write  $g(u) = h(\pi(u)) g'(u)$  and differentiate w.r.t.  $u$  (this is essentially the tangent map), we get 2 terms from a chain rule. One term comes from writing

$$g(u) = h(\pi(u)) g'(u) = L_{h(x)} g'(u),$$

holding  $x = \pi(u)$  fixed and differentiating w.r.t. the  $u$  in  $g'(u)$ , to get

$$L_{h(x)} g'_*$$

and the second from writing

$$g(u) = h(\pi(u)) g'(u) = R_{g'(u)} h(\pi(u)) = R_{a'} h(\pi(u))$$

and differentiating w.r.t. the  $u$  in  $\pi(u)$  while holding  $a' = g'(u)$  fixed.

This gives

$$R_{a'} h_* \pi_*$$

Adding the two together gives the total derivative,

$$g_* = L_{h(x)} g'_* + R_{a'} h_* \pi_*$$

So the 2nd term in  $\omega|_u$  is

~~$$R_{a'} h_* \pi_*$$~~

$$L_{a^{-1}} g_* =$$

$$L_{a^{-1}} L_{h(x)^{-1}} [L_{h(x)} g'_* + R_{a'} h_* \pi_*]$$

$$= L_{a^{-1}} g'_* + A_{a^{-1}} L_{h(x)^{-1}} h_* \pi_*$$

where we use  $A_{a^{-1}} = L_{a^{-1}} R_{a'}$  and the fact that left and right translations commute. Altogether we have

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$$\begin{aligned}
 \omega|_u &= \text{Ad}_{a^{-1}}(\pi^*A)|_u + L_{a^{-1}*}g^*|_u \\
 &= \text{Ad}_{a^{-1}}\left(\text{Ad}_{h(x)^{-1}}(\pi^*A)|_u + L_{h(x)^{-1}*}h^*\pi|_{x_u}\right) + L_{a^{-1}*}g^*|_u \\
 &= \text{Ad}_{a^{-1}}(\pi^*A')|_u + L_{a^{-1}*}g^*|_u
 \end{aligned}$$

if we set

$$A' = \text{Ad}_{h(x)^{-1}}A + L_{h(x)^{-1}*}h^*$$

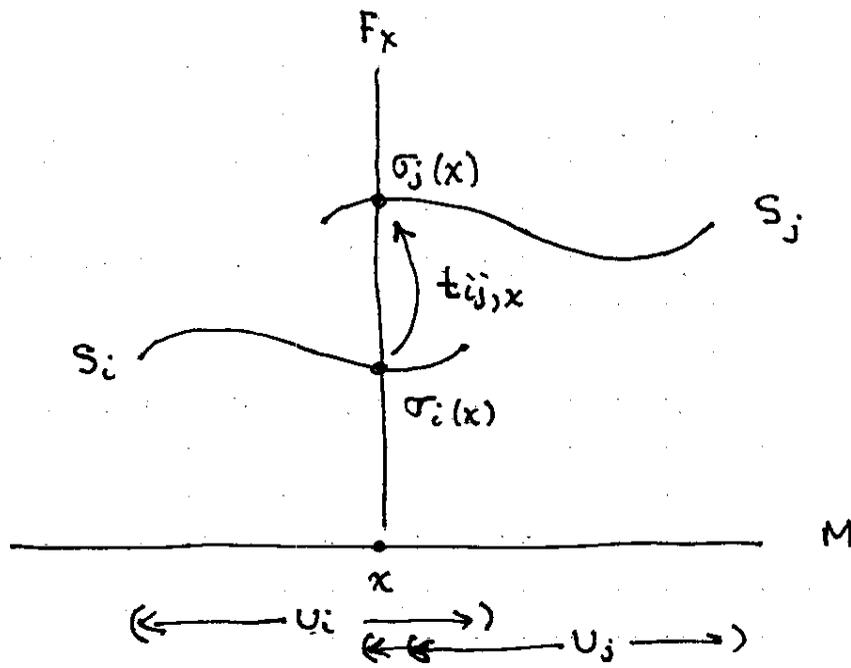
Nakahara would write this as

~~$A' = h^{-1}Ah + h^{-1}dh$~~

$$A' = h^{-1}Ah + h^{-1}dh$$

Showing how  $A$  transforms under a gauge transformation over a single chart  $U$  through the function  $h: U \rightarrow G$ .

Transforming  $A$  between charts  $U_i$  and  $U_j$  is the same calculation except for a change of notation. Here is the picture:



The two sections are

$$\sigma_i : U_i \rightarrow P$$

$$\sigma_j : U_j \rightarrow P$$

and they are related by

$$\sigma_j(x) = \sigma_i(x) t_{ij,x} \quad , \quad x \in U_i \cap U_j$$

where  $t_{ij} : U_i \cap U_j \rightarrow G$  is the transition function, which takes the place of  $h$  in the previous calculation. Thus we have

$$A_j = t_{ij}^{-1} A_i t_{ij} + t_{ij}^{-1} dt_{ij}$$

So, to specify a connection, we need a set  $\{(U_i, A_i, \sigma_i)\}$  where  $\{U_i\}$  is an open cover of  $M$ , and

$$A_i \in \mathfrak{g} \otimes \Omega^1(U_i),$$

$$\sigma_i : U_i \rightarrow P \quad \text{a local section.}$$

This is a 4th way of specifying a connection.

short  
End of the semester