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We get another phase convention if we multiply our original spinor by $e^{-i\varphi/2}$, giving

$$z_S(\theta, \varphi) = \begin{pmatrix} e^{-i\varphi} \cos \theta/2 \\ \sin \theta/2 \end{pmatrix},$$

which is smooth over the southern hemisphere but has a singularity at the north pole.

Our inability to find a phase convention that is smooth everywhere over S^2 is a sign that the Hopf bundle is nontrivial. But we can cover S^2 with two open sets U_N and U_S , as in the monopole vector potential, and we have the open cover necessary to make the Hopf bundle into a bundle according to the official definition. The spinors z_N and z_S defined above are local sections of the bundle, which imply local trivializations ϕ_N and ϕ_S .

Explicitly,

$$\begin{aligned} \phi_N: U_N \times G &\rightarrow \pi^{-1}(U_N) \\ : ((\theta, \varphi), e^{i\alpha}) &\mapsto e^{i\alpha} z_N(\theta, \varphi) = e^{i\alpha} \begin{pmatrix} \cos \theta/2 \\ e^{i\varphi} \sin \theta/2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \phi_S: U_S \times G &\rightarrow \pi^{-1}(U_S) \\ : ((\theta, \varphi), e^{i\alpha}) &\mapsto e^{i\alpha} z_S(\theta, \varphi) = e^{i\alpha} \begin{pmatrix} e^{-i\varphi} \cos \theta/2 \\ \sin \theta/2 \end{pmatrix}. \end{aligned}$$

There is only one transition function,

$$t_{NS, x} = \phi_{N, x}^{-1} \phi_{S, x}$$

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or

$$\phi_{S,x} = \phi_{N,x} t_{NS,x}$$

where $x \in U_N \cap U_S$, i.e., the equator where $\theta = \pi/2$, so write $x = (\frac{\pi}{2}, \varphi)$ or just φ . Then putting $\theta = \pi/2$, we get

$$\frac{e^{i\alpha}}{\sqrt{2}} \begin{pmatrix} e^{-i\varphi} \\ 1 \end{pmatrix} = \frac{e^{i\alpha}}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\varphi} \end{pmatrix} t_{NS}(\varphi),$$

or

$$t_{NS}(\varphi) = e^{-i\varphi}.$$

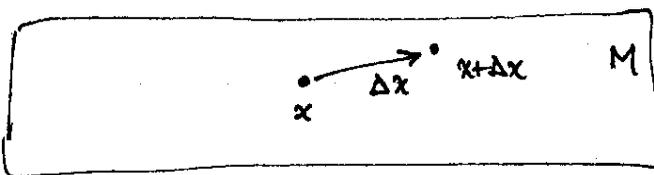
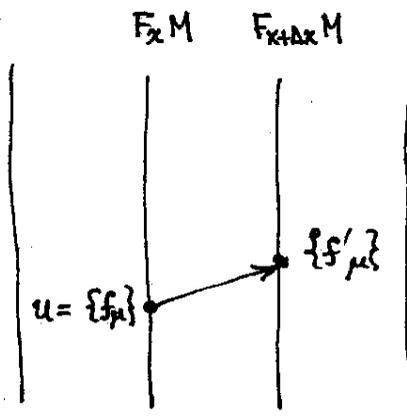
The transition functions belong to the homotopy class $n = -1$ in $\pi_1(S^1)$. Thus the Hopf bundle is not trivial, as we suspected.

Now we turn to the subject of connections. The example we know of connections concerned the parallel transports of tangent vectors, i.e., in TM. The problem of parallel transport occurs in other vector bundles, too, for example the H.L.B. on which ψ (quantum state) lives. Without a definition of parallel transport (a connection) it is impossible to define a derivative (covariant deriv) operation, and thus impossible to have differential equations (field equations) that have a geometrical meaning. For example, a Schrödinger equation for ψ . The natural setting for understanding connections is a P.F.B. It turns out connections on a P.F.B. are equivalent to gauge potentials (vector potentials) on the base space.

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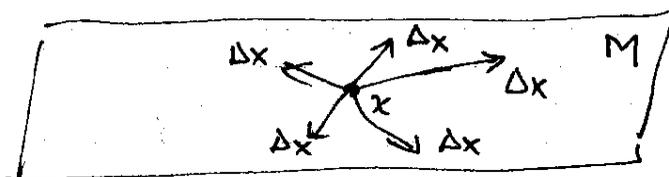
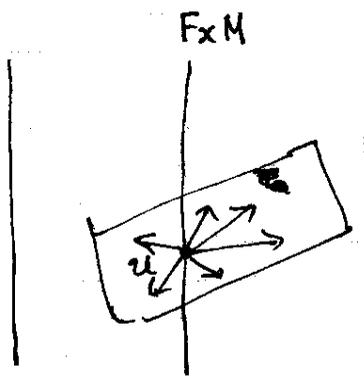
Begin with the example we know, connections on TM . we introduced these originally for defining infinitesimal parallel transport of tangent vectors, i.e., creating an identification of nearby tangent spaces $T_x M$ and $T_{x+\Delta x} M$. Now let's view this in the frame bundle FM .

Let $\{f_\mu\}$ be a specific frame at $x \in M$, i.e., a point in the frame space $F_x M$ over x . This is not a field, just one frame. By parallel transporting each of the vectors that makes up the frame (f_μ) over to a nearby tangent space, we get a



new frame, call it $\{f'_\mu\}$. This creates a small displacement vector in the bundle (betw. $\{f_\mu\}$ and $\{f'_\mu\}$) corresponding to a given small displacement vector Δx in the base space. If we let Δx range over all nearby tangent spaces, the corresponding vector in the bundle sweeps out a surface,

since the vectors are infinitesimal, this surface lies in $T_u P$, where $u = \{f_\mu\}$ = the original given frame and $P = FM$ (a P.F.B.).



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Thus we get a map: $T_x M \rightarrow T_u P$, where $u \in F_x M$, defined for any $u \in F_x M$, once we have a connection. This map is linear, because \parallel transport is linear in Δx , so it specifies a vector subspace of $T_u P$. Notice the dimensions,

$$\dim T_x M = \dim M$$

$$\dim T_u P = \dim P = \dim M + \dim G.$$

The structure group $G = GL(m, \mathbb{R})$ or maybe $SO(m)$ or some other group. So the map maps a smaller vector space to a larger one. The kernel of the map is $\{0\}$, since if Δx is nonzero, then it connects two distinct points of M , and the corresponding vector in $T_u P$ must be nonzero, because it joins two distinct fibers (over x and $x + \Delta x$). Therefore $\dim \text{im}(\text{map}) = \dim M$, the map has maximal rank. The subspace of $T_u P$ swept out by the process described is isomorphic to $T_x M$.

The subspace described ($\text{im } T_x M$ under our map) is called the horizontal subspace at $u \in P$, denoted $H_u P$.

One can also define a vertical subspace $V_u P$ at $u \in P$. It is just the tangent space ~~at~~ at u to the fiber,

$$V_u P = T_u(F_x), \quad \text{where } x = \pi(u).$$

Since $\dim F_x = \dim G$, we have

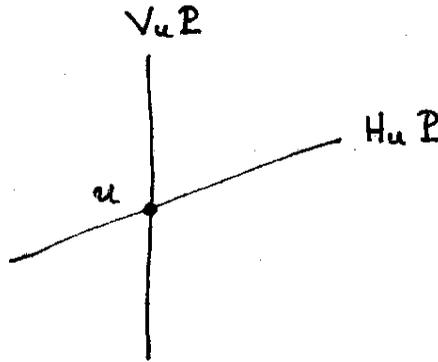
$$\dim V_u P = \dim G.$$

Any bundle has a vertical subspace, but we get a horizontal subspace on a P.F.B. only if there is a connection.

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We sketch it like this,



at some angle because we don't wish to imply that $H_u P$ is orthogonal to $V_u P$ (in general there is no metric on P , even if there is one on the base space M). However, $H_u P$ is transverse to $V_u P$, meaning

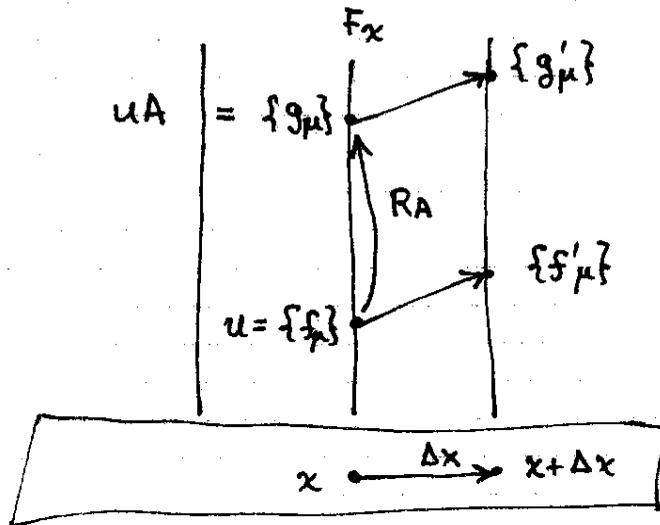
$$T_u P = H_u P \oplus V_u P$$

i.e., any vector in $T_u P$ can be represented uniquely as the sum of a horizontal and a vertical vector.

The horizontal subspaces have another property which we see if we parallel transport another frame $\{g_\mu\} \in F_x M$, where

$$g'_\mu = f_\nu A^\nu{}_\mu, \quad A \in G \quad (A = \text{some matrix}).$$

This is the right action of G on P in the case of a frame bundle; we can write R_A for this action. The picture:



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What is the relation between $\{f'_\mu\}$ and $\{g'_\mu\}$? Because
 if transport is linear in the vector transported, we have

$$g'_\mu = f'_\nu A^\nu{}_\mu$$

(the parallel transport does not affect the matrix A). That is,

$$\{g'_\mu\} = R_A \{f'_\mu\}$$

if $\{g_\mu\} = R_A \{f_\mu\}$.

Parallel transport and right action commute. Thus, if
 ~~$X \in \mathfrak{H}_u P$, then $R_{a*} X \in \mathfrak{H}_{ua} P$~~ $Y \in \mathfrak{H}_u P$, then $R_{a*} Y \in \mathfrak{H}_{ua} P$.

In fact, since R_a is a diffeomorphism (R_{a*} has full rank),
 we have

$$R_{a*} (\mathfrak{H}_u P) = \mathfrak{H}_{ua} P.$$

We now define a connection on a P.F.B. (P, M, G, π)
 as a smooth assignment of ^{"horiz."} subspaces $\mathfrak{H}_u P \subset T_u P$ at each
 $u \in P$ such that

$$(a) \quad T_u P = \mathfrak{H}_u P \oplus \mathfrak{V}_u P$$

$$(b) \quad R_{a*} (\mathfrak{H}_u P) = \mathfrak{H}_{ua} P.$$

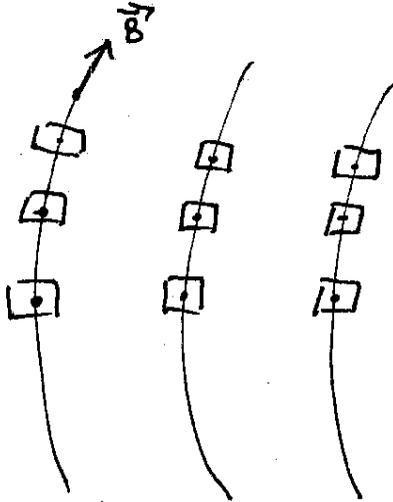
(General defn for any P.F.B. Later we give another, equivalent
 defn.)

Now we make a digression. On any manifold M ,
 we define an r-distribution Δ as a smooth assignment of
 r -dimensional subspaces $(0 \leq r \leq \dim M)$ in each tangent space
 $T_x M$.

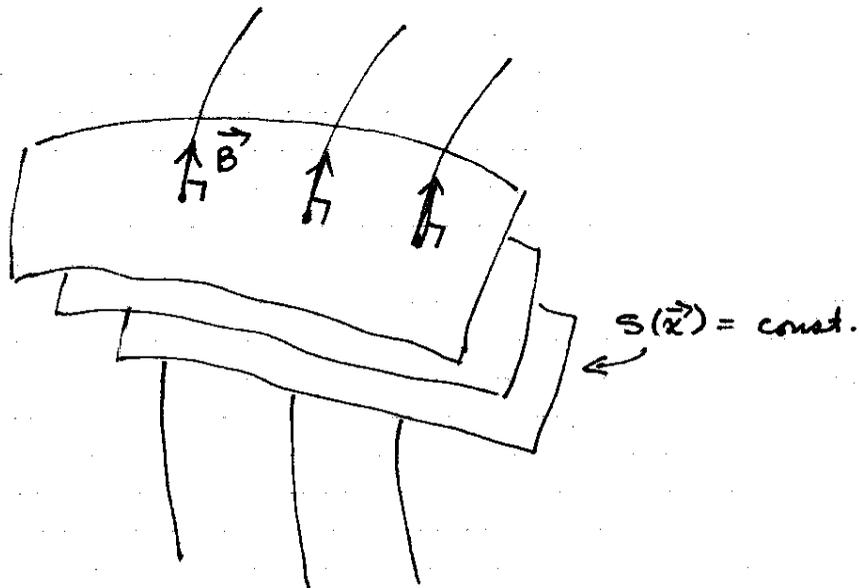
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For example, consider $M \subseteq \mathbb{R}^3$, a region in which a nonzero magnetic field \vec{B} is defined. The field has field lines (the integral curves of \vec{B}). At each point of this region we consider the planes (or small pieces of planes) \perp to \vec{B} .



The planes really live in the tangent spaces. A natural question is whether these little pieces of planes can be glued together smoothly to form surfaces (actually, a family of surfaces). If so, \vec{B} is everywhere \perp to the surfaces.



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By assigning values to these surfaces (const on each surface), we can regard the surfaces as level sets of a scalar $s(\vec{x})$. Then since $\vec{B} \perp$ surfaces, we have

$$\vec{B} = \mu \nabla s$$

where μ is some scalar (it can depend on \vec{x}). But this implies

$$\nabla \times \vec{B} = \nabla \mu \times \nabla s,$$

$$\vec{B} \cdot (\nabla \times \vec{B}) = 0.$$

So such surfaces exist only if $\vec{B} \cdot \nabla \times \vec{B} = 0$. In fact, it is iff, $\vec{B} \cdot \nabla \times \vec{B} = 0$ is the integrability condition for the existence of scalars μ and s in $\vec{B} = \mu \nabla s$, given \vec{B} . We see that in general, 2D surfaces orthogonal to a vector field in 3D do not exist.

We say that an r -distribution Δ on M is integrable if there exists (locally) a foliation of M into r -dimensional submanifolds such that at each x (in some local region of M) Δ_x is tangent to the manifold passing through x .

We say that a vector field $X \in \mathcal{X}(M)$ lies in a distribution Δ if $X|_x \in \Delta|_x$ at each x . If Δ is integrable, then ^{any} ~~the~~ integral curves of $X \in \Delta$ must lie in one of the submanifolds to which Δ is tangent (a fairly obvious geometrical fact). Thus, by following integral curves of vector fields lying in an integrable distribution Δ , we can explore the corresponding submanifolds. ~~Some~~ Such vector fields need not commute, but if the distribution is

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integrable, then $[X, Y]$ should lie in Δ . (Another fairly intuitive fact.)
 These ideas make plausible the following theorem:

Thm. A distribution Δ is integrable iff $X, Y \in \Delta$ (X, Y vector fields on M) implies

$$[X, Y] \in \Delta.$$

This is the Frobenius theorem. It has several versions. If we specify an r -distribution by r linearly indep. vector fields X_1, \dots, X_r , then Δ is integrable iff

$$[X_i, X_j] = c_{ij}^k X_k,$$

where the c_{ij}^k are allowed to be functions of position. (Thus, the $\{X_i\}$ do not usually form a Lie algebra.)

To return to connections, we can say that a connection on a P.F.B. is an m -dimensional distribution on P ($m = \dim M$), invariant under the group action and transverse to the fibers.

In general, this distribution which defines a connection is not integrable. One can show that it is integrable iff the curvature tensor (on M) vanishes.

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We now develop an alternative way to define a connection. It is related to the idea that a vector subspace (like $H_u P$) can be specified by the forms that annihilate it. Let us sketch $T_u P$, with horiz. and vertical subspaces: An arbitrary vector $Y \in T_u P$ can be uniquely decomposed into horiz. and vert. components,

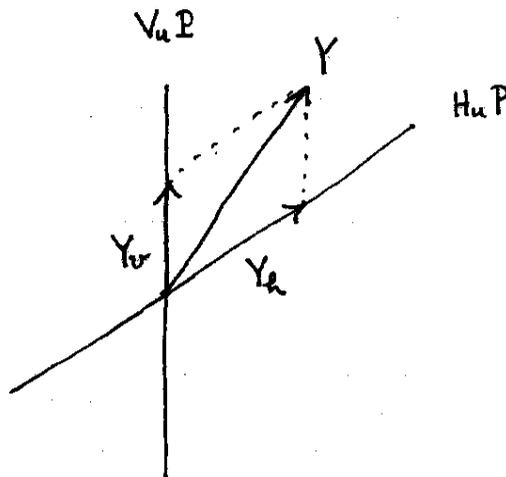
$$Y = Y_v + Y_h$$

Define projection operators Π_h, Π_v such that

$$Y_v = \Pi_v(Y),$$

$$Y_h = \Pi_h(Y),$$

$$\Pi_v + \Pi_h = 1.$$



The vertical projector is a map $\Pi_v: T_u P \rightarrow V_u P$. It has the property that $H_u P = \ker \Pi_v$.

The vertical space $V_u P$ is isomorphic with the Lie algebra \mathfrak{g} of G . This is true for all u (a different isomorphism for each u). This comes about because of the (right) action of G on P , $a \mapsto R_a$, $a \in G$, $R_a: P \rightarrow P$. Actually, since R_a preserves fibers, we can think $R_a: F_x \rightarrow F_x$. This action induces vector fields on F_x , which are purely vertical (as seen within P). ~~the~~

In the following it is convenient to denote a vector at a point as an equivalence class of curves, say, $X = [\sigma(t)]$ where $\sigma(0) = x$

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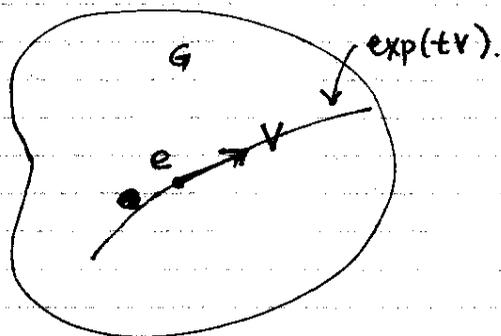
and $X \in T_x M$. This notation makes it convenient to apply the tangent map f_* to the vector (where say $f: M \rightarrow N$), according to

$$f_*[\sigma'(t)] = [f\sigma'(t)].$$

It is a convenient way of dealing with tangent maps in coordinate-free notation.

For example, let $V \in \mathfrak{g}$. A curve passing through e at $t=0$ with tangent vector V at $t=0$ is $\exp(tV)$ (the integral curve of the left- or right-invariant vector field ~~associated~~ associated with V). That is,

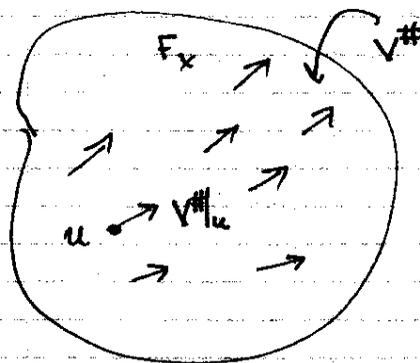
$$V = [\exp'(tV)].$$



Letting G act on F_x by the right action R_a , we get an induced vector field $V^\#$ on F_x , corresponding to $V \in \mathfrak{g}$. Note that V is just one vector, while $V^\#$ is a vector field.

$V^\#$ evaluated at a point $u \in F_x$ can be seen as an equivalence class of curves,

$$\begin{aligned} V^\#|_u &= [R_{\exp(tV)} u] \\ &= [u \exp(tV)]. \end{aligned}$$



This is the definition of $V^\#$ (equivalent to the one given earlier, but note that earlier we used left actions instead of right actions.)

Using this construction we get a map: $\mathfrak{g} \rightarrow T_u(F_x): V \mapsto V^\#|_u$.

This map is invertible (it is a vector space isomorphism) because the right action of R_a on F_x is free. We can call this map $\#$ or $\#_0$.

To go back to the vertical projector Π_v , we can construct a chain of maps,

$$T_u P \xrightarrow{\Pi_v} V_u P \xrightarrow{\#^{-1}} \mathfrak{g}.$$

Composing these, we get the definition of a Lie algebra-valued 1-form on P , called the Ehresmann form:

Def: $\omega = \#^{-1} \circ \Pi_v$

$$\omega|_u : T_u P \rightarrow \mathfrak{g}$$

$$\omega \in \mathfrak{g} \otimes \Omega^1(P).$$

The Ehresmann form has the following properties. First, it annihilates the horizontal subspace, because Π_v does so:

$$\omega|_u (H_u P) = 0.$$

Second, if we let ω act on an arbitrary ~~and~~ vertical vector, write it $V^\#|_u$ for some $V \in \mathfrak{g}$, then the Π_v does nothing and $\#^{-1}$ strips off the $\#$:

$$\omega|_u (V^\#|_u) = V, \quad \forall V \in \mathfrak{g}.$$

Third, ω has a certain behavior when pulled back by R_a . Consider

$$(R_a^* \omega)|_u (V^\#|_u) = \omega|_{ua} (R_{a*} (V^\#|_u)),$$

where we evaluate on an arbitrary vertical vector and use the defn. of

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the pull-back. Now, from above we have

$$\begin{aligned} R_{a*}(V^\#|_u) &= R_{a*}[u \exp(tV)] \\ &= [R_{a*} u \exp(tV)] \\ &= [u \exp(tV) a] \\ &= [ua a^{-1} \exp(tV) a] \end{aligned}$$

Now we use the identity,

$$a^{-1} \exp(tV) a = \exp(t \operatorname{Ad}_{a^{-1}} V), \quad (\text{more on this identity below}).$$

where $g \mapsto \operatorname{Ad}_g$ is the adjoint representation of G . Thus,

$$\begin{aligned} &\rightarrow = [ua \exp(t \operatorname{Ad}_{a^{-1}} V)] \\ &= (\operatorname{Ad}_{a^{-1}} V)^\#|_{ua}. \end{aligned}$$

So,

$$\begin{aligned} (R_a^* \omega)|_u (V^\#|_u) &= \omega|_{ua} ((\operatorname{Ad}_{a^{-1}} V)^\#|_{ua}) \\ &= \operatorname{Ad}_{a^{-1}} V \\ &= \operatorname{Ad}_{a^{-1}} \omega|_u (V^\#|_u). \end{aligned}$$

Thus, two operators, $(R_a^* \omega)|_u$ and $\operatorname{Ad}_{a^{-1}} \omega|_u$, have the same action on arbitrary vertical vectors. Let's see what they do to horizontal vectors:

$$\begin{aligned} (R_a^* \omega)|_u (H_u P) &= \omega|_{ua} (R_{a*}(H_u P)) \\ &= \omega|_{ua} (H_{ua} P) = 0 \end{aligned}$$

$$\operatorname{Ad}_{a^{-1}} \omega|_u (H_u P) = 0.$$

So they have the same action on all vectors, the operators are equal, and we have $R_a^* \omega = \operatorname{Ad}_a \omega$. To summarize the properties of the

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Ehresmann form, we have:

$$\begin{aligned} (1) \quad \omega|_u(H_u P) &= 0 \\ (2) \quad \omega|_u(V^\#|_u) &= V \\ (3) \quad R_a^* \omega &= \text{Ad}_{a^{-1}} \omega \end{aligned}$$

Digression on the identity above. Let $g \in G$, $V \in \mathfrak{g}$. In the case of matrix groups, g and V are matrices, and $\text{Ad}_g V = gVg^{-1}$. Then the identity,

$$g e^{tV} g^{-1} = e^{t(gVg^{-1})}$$

follows immediately from power series. (For matrix groups, $\exp(tV)$ is a genuine exponential.) For arbitrary Lie groups, $\text{Ad}_g = \text{I}_g * \text{I}_e$, where $\text{I}_g: G \rightarrow G: a \mapsto gag^{-1}$ is the inner automorphism action of G on itself. Thus, $\text{Ad}_g = \text{L}_g * \text{R}_{g^{-1}} * \text{I}_e$, since $\text{I}_g = \text{L}_g \text{R}_{g^{-1}}$. $\text{Ad}_g: \mathfrak{g} \rightarrow \mathfrak{g}$ is the action of G on its own Lie algebra, the adjoint representation. Now note that $g \exp(tV) g^{-1}$ is a one-parameter subgroup,

$$\begin{aligned} (g \exp(sV) g^{-1})(g \exp(tV) g^{-1}) &= g \exp(sV) \exp(tV) g^{-1} \\ &= g \exp((s+t)V) g^{-1}. \end{aligned}$$

So it must have the form $\exp tW$, for some $W \in \mathfrak{g}$. To find out what W is, compute the tangent vector at the identity,

$$\begin{aligned} W &= [\exp(tW)] = [g \exp(tV) g^{-1}] = \text{L}_g * \text{R}_{g^{-1}} * [\exp tV] \\ &= \text{Ad}_g V. \end{aligned}$$