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is just the probability density there. However, there are several subtleties to the problem of measuring the phase of ψ.

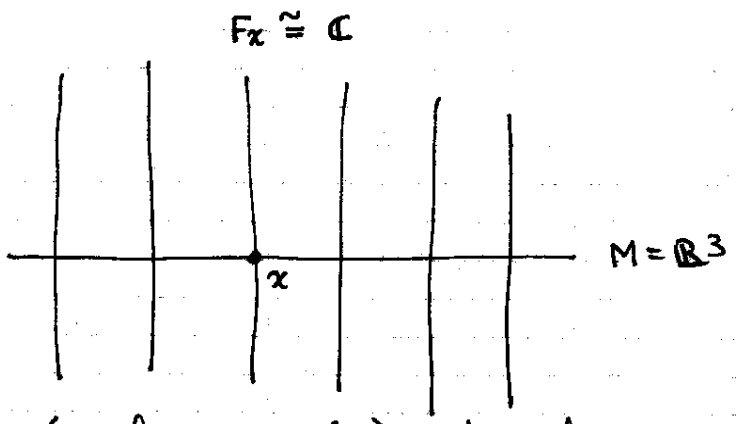
One is that the phase of ψ is not even determined until we specify a convention for the vector potential. This is because when we do a gauge transformation on $\vec{A}$, ψ changes also:

$$\vec{A}' = \vec{A} + \nabla\chi \qquad \chi = \text{"gauge scalar"}$$

$$\psi'(\vec{x}) = e^{-\frac{ie}{\hbar c}\chi(\vec{x})}\,\psi(\vec{x}).$$

ψ changes by a phase factor that depends on $\vec{x}$, thus the phase of ψ cannot be definite without a definite convention for $\vec{A}$.

This situation is clarified if we introduce a bundle with $M = \mathbb{R}^3$ (physical space), and $F = \mathbb{C}$. This is an example of a <u>Hermitian line bundle</u>. A line bundle in general is any vector bundle in which the fibers are 1D vector spaces (real or complex). Here they are complex. The "Hermitian" part means that we introduce a complex metric onto each fiber, that allows us to compute $|\psi|^2$ at each point x. Since each fiber has a metric, we restrict consideration to orthonormal frames, so the structure group is $U(1) = G$. (We would use $U(n)$ if $F = \mathbb{C}^n$.)



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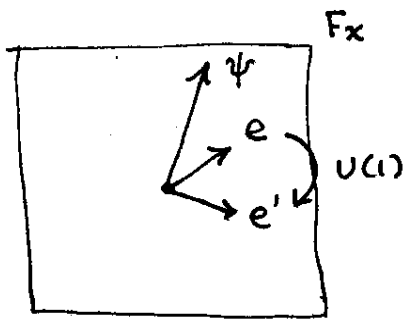
We ignore the possibility that the individual fibers F_x might have a natural isomorphism with $F = \mathbb{C}$, since on a general bundle there would be none, and think of F_x as a plane (a 1D complex vector space) with an origin (a 0-vector) but no preferred basis. A basis would consist of one nonzero vector in F_x, call it e. We choose it to be a unit vector.

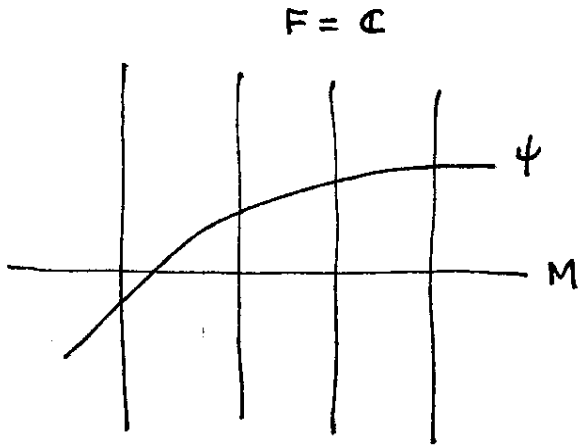


Given the basis e, an arbitrary vector in F_x, call it ψ, can be represented as a linear combination of e with a complex coefficient, call it $\psi(x)$:

$$\underset{\substack{\uparrow \\ \text{abstract} \\ \text{vector}}}{\psi} = e \; \underset{\substack{\uparrow \\ \text{its component} \\ \text{w.r.t. } e}}{\psi(x)}$$

We can of course change basis, $e \to e'$ by a $U(1)$ transformation, whereupon the vector ψ acquires a new component $\psi'(x)$.

We interpret the equation for the transformation of ψ under a gauge transformation as giving two components, $\psi(x)$ and $\psi'(x) \in \mathbb{C}$, of one abstract vector ψ w.r.t two different bases e and e', related by the x-dependent $U(1)$ phase factor $e^{-\frac{ie}{\hbar c}\chi(\vec{x})}$. Then the quantum state of the particle is represented by a section of the Hermitian line bundle,



This is a global section, since ψ should exist everywhere. This is a pleasing picture, because the section does not depend on the gauge.

We must suppose somehow that a field of frames in the frame bundle, called e or e' above, is specified once a gauge convention has been chosen for $\vec{A}$. The precise manner in which this occurs will be explained later. We just note for now that a section of the frame bundle is in general only locally defined, since a global section of the frame bundle exists iff the frame bundle is trivial. Thus, the wave function $\psi(x)$ will be only locally defined, in general, even though the section ψ is global.

In the case that $M = \mathbb{R}^3$ (or $\mathbb{R}^4$ for a time-dependent problem), all the bundles discussed above are trivial, since $\mathbb{R}^3$ or $\mathbb{R}^4$ is contractible. In the case of a trivial bundle, the global section ψ can be described by a global wavefunction $\psi(x)$, and we return to the usual way of thinking about wave functions, as complex-valued fields,

$$\psi : \mathbb{R}^3 \to \mathbb{C} .$$

A <u>field</u> in the usual sense is just a mapping,

$$\psi : M \to F$$

where usually $M \subseteq \mathbb{R}^3$ or $\mathbb{R}^4$ and F is the "field value space",

for example, $\mathbb{R}^3$ for an electric field $\vec{E}$. Thus a field in this case is just a special case of a map between two spaces. Write such a map generally as $f: A \to B$. The map f can be seen geometrically as the set of points $(a, b) \in A \times B$ such that $b = f(a)$. This is just the graph of the function, but it can also be seen as a global section of the trivial fiber bundle $A \times B$. Thus, a global section of a possibly nontrivial fiber bundle is a generalization of the usual notion of a function or map between two spaces, or of a field in physical application.

An application where the Hermitian line bundle carrying the section ψ is nontrivial occurs when we have a charged particle moving in the field of a magnetic monopole,

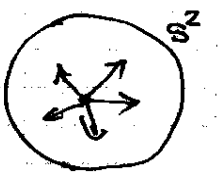
$$\vec{B} = g \frac{\hat{r}}{r^2}.$$

We remove the origin to avoid the singularity, and set $M = \mathbb{R}^3 - \{0\}$. This is not a contractible space. The magnetic field 2-form is

$$B = \frac{1}{2} \varepsilon_{ijk} B_i \, dx^j \wedge dx^k,$$

it satisfies $dB = 0$, which is equivalent to $\nabla \cdot \vec{B} = 0$. But $B \neq dA$, i.e., there does not exist a global, smooth vector potential for $\vec{B}$. This is because a closed form is exact iff its integral over all cycles vanishes. But letting S^2 be any sphere surrounding the monopole, we have

$$\int_{S^2} B = 4\pi g,$$

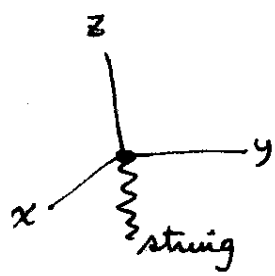


so B is not exact.

If we write $\vec{B} = \nabla \times \vec{A}$ in spherical coordinates, it is easy to uncurl $\vec{B}$ and find $\vec{A}$. The solution is not unique, but one that emerges is

$$\vec{A}_N = g \frac{(1-\cos\theta)}{r\sin\theta}\hat{\varphi} = g\frac{\tan\theta/2}{r}\hat{\varphi}$$

This $\vec{A}$ is singular on the negative z-axis, where $\tan\theta/2 \to \infty$ and $\hat{\varphi}$ is undefined. But it is smooth on the positive z-axis, since $\tan\theta/2 \to 0$ and the lack of direction of $\hat{\varphi}$ is not a problem. The (half) line of singularities of $\vec{A}$ (the neg. z-axis) is called the <u>string</u> of the monopole, although it should be called the string of the vector potential.

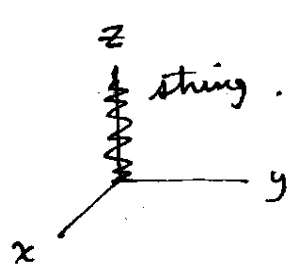


Since this $\vec{A}$ is smooth and nonsingular over the northern hemisphere, we write $\vec{A}_N$ for it.

We can get an $\vec{A}$ that is nonsingular over the southern hemisphere by uncurling $\vec{B}$ in a slightly different manner. This gives

$$\vec{A}_S = g\frac{(-1-\cos\theta)}{r\sin\theta}\hat{\varphi} = -g\frac{\cot\theta/2}{r}\hat{\varphi}.$$

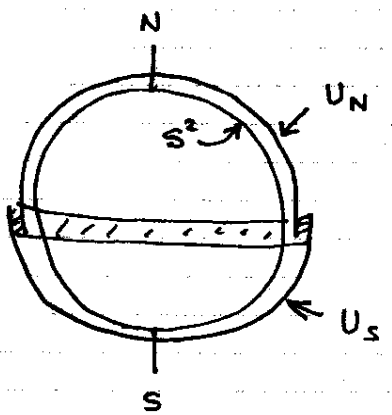
It has a string on the positive z-axis.



We see that it is necessary to cover $M = \mathbb{R}^3 - \{0\}$ with at least two charts, with $\vec{A}$ smooth in each chart. This suggests that the $U(1)$ P.F.B. for the charged particle in the monopole field is nontrivial (since a gauge convention somehow is related to a section of this PFB, and apparently this PFB does not have a global section).

The above is motivation to consider $U(1)$ bundles over $M = \mathbb{R}^3 - \{0\}$. Actually, it suffices to study $U(1)$ bundles over S^2, since the latter is a deformation retract of $\mathbb{R}^3 - \{0\}$. How many (inequivalent) circle bundles are there over S^2?

We can cover S^2 with two open sets, U_N which covers the northern hemisphere and extends slightly south of the equator, and U_S which does something similar from the southern side. These overlap slightly at the equator. If we have any bundle over S^2, then ~~this~~ the part of this bundle over U_N or U_S is trivial, since these are contractible. Thus there exist local trivializations ϕ_N and ϕ_S over these regions.



The overlap $U_N \cap U_S$ is a small strip around the equator, where φ (the azimuthal angle) is a coordinate. The transition function $t_{NS} : U_N \cap U_S \to G = U(1)$ is therefore a function of φ, and we can write

$$t_{NS}(\varphi) = e^{i\alpha} \in U(1)$$

where $\alpha = \alpha(\varphi)$. We note that t_{NS}: equator $\to U(1)$, i.e., $t_{NS}: S^1 \to S^1$, so t_{NS} (as a map) belongs to some homotopy class in $\pi_1(S^1) = \mathbb{Z}$. The class is characterized by an integer $n \in \mathbb{Z}$.

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Now consider a "local gauge transformation" defined by functions $g_N : U_N \to G$, $g_S : U_S \to G$, taking old transition functions to new ones according to

$$\tilde{\phi}_{N,x} = \phi_{N,x} \circ g_N(x)$$

$$\tilde{\phi}_{S,x} = \phi_{S,x} \circ g_S(x)$$

where $x \in U_N$ or U_S. The new transition functions are

$$\tilde{t}_{NS,x} = g_N(x)^{-1}\, t_{NS,x}\, g_S(x),$$

where now $x \in U_N \cap U_S$, i.e., $x \in$ equatorial strip. So we can replace x by φ in this formula if we want. To what extent can we simplify (or perhaps even transform away) the transition functions by such a gauge transformation? First note, all 3 factors above are elements of $U(1)$, an Abelian group, so we can write them in any order. Thus there is no loss of generality in letting $g_S(x) = e$ and letting $g_N(x)$ do all the work.

Look at U_N from above, where it is like a disk.

Examine the behavior of $g_N(x)$ as x goes around in a circle about the north pole. This gives a map: $S^1 \to U(1) = S^1$, which has a homotopy class $\in \pi_1(S^1) = \mathbb{Z}$. When the circle is very small, $g_N(x)$ is nearly constant, so the homotopy class is 0. By continuity, this must remain the homotopy class as the circle enlarges to the equator. Therefore on the equator, where we have

$$\tilde{t}_{NS}(\varphi) = g_N(\varphi)^{-1}\, t_{NS}(\varphi),$$

the homotopy class of $\tilde{t}_{NS}$ must be the same as that of t_{NS}.

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We can change $t_{NS}(\varphi)$ into a any new function of the same homotopy class by means of a gauge transformation, but we cannot change it to any other homotopy class. In particular, if $n \neq 0$, we cannot gauge away t_{NS}, and the bundle is nontrivial.

We see that $U(1)$ bundles over S^2 are characterized by integers $\in \pi_1(S^1)$. There is an infinite but discrete set of such bundles.

The above analysis leaves two questions unanswered:

① How precisely is a choice of gauge for $\vec{A}$ linked to a choice of section in the PFB associated with the HLB carrying ψ for the charged particle?

② Since there are an ∞ number of $U(1)$ bundles over S^2, how do we know which is the right one for ψ?

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Start today by computing the index (homotopy class of $\pi_1(S^1)$ of the transition function) of a specific $U(1)$ bundle over S^2, namely, the Hopf ~~map~~ fibration. The summary is the following:

$$E = P = S^3 \qquad \text{(principal fiber bundle)}$$
$$M = S^2$$
$$F = G = U(1)$$

~~Right action of G on P~~ Let $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ be a 2-component spinor, $z \in \mathbb{C}^2 = \mathbb{R}^4$, or if normalized, $|z|^2 = 1$, then $z \in S^3 \subset \mathbb{R}^4$, as we henceforth assume. The (right) action of G on P ($U(1)$ on S^3) is

$$z \mapsto e^{i\alpha} z, \qquad z \in S^3, \quad e^{i\alpha} \in U(1).$$

It's just changing the "overall phase" of the spinor. It's conventionally taken to be a right action, but since $U(1)$ is Abelian you can apply it in any order. The projection $\pi : S^3 \to S^3/U(1) = S^2$ is defined by

$$\pi(z) = \langle z | \vec{\sigma} | z \rangle = \text{"the direction the spinor}$$

is pointing in", to use standard QM language. Call this $\hat{n}(z) \in S^2$. It's a unit vector. Use coordinates (θ, φ) on S^2.

To compute the homotopy class we need transition functions, and to compute those we need local trivializations ϕ_i or $\phi_{i,x}$. Notation:

$$\phi = \text{local trivialization}$$
$$\varphi = \text{azim. angle on sphere.}$$

since there are two ~~phi~~ phis.

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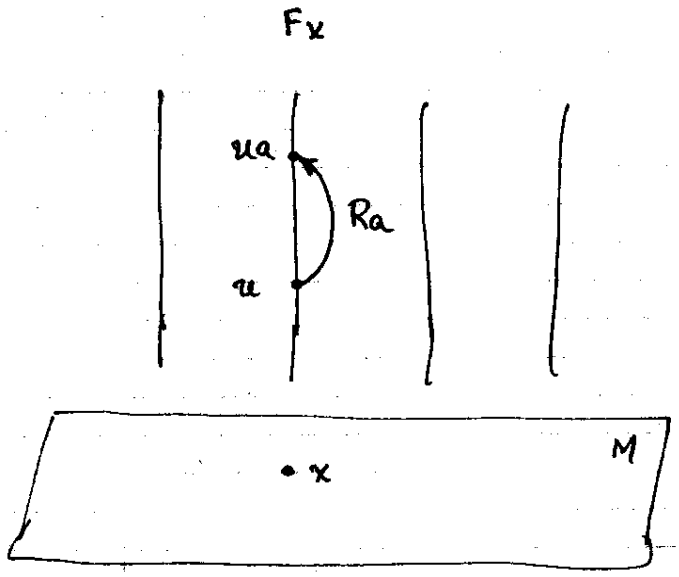
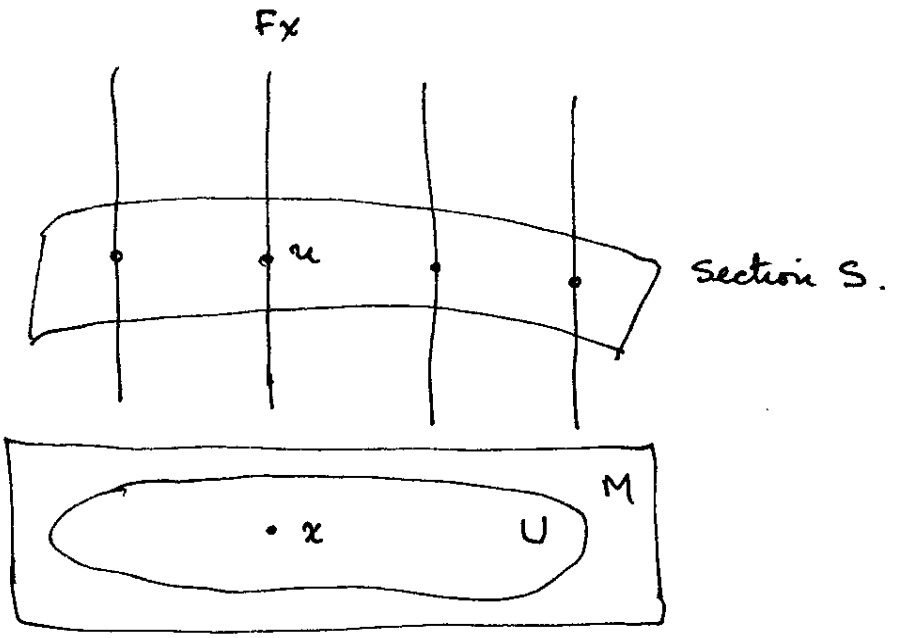
About local triv's in a P.F.B. (in general). The fibers are diffeo to the group G, but unlike a group they do not have an identity element. But if we take a fiber (over x, say) and arbitrarily ~~use~~ pick one point $u \in F_x$ as a reference point, then all other points of that fiber are accessible (and parameterizable) by means of the group element needed to reach them by the right ~~multi~~ action of G on P.



We denote the right action by $a \mapsto R_a$, where $a \in G$ and $R_a : P \to P$. It satisfies $R_a R_b = R_{ba}$, and $\pi R_a u = \pi u$ (the action is fiber preserving). We denote $R_a u$ by ua for short, where $u \in P$. In the picture above, point u on fiber F_x is identified with $e \in G$ and ua with $a \in G$. This gives us a map on one fiber,

$$\phi_{i,x} : G \longrightarrow F_x : a \longmapsto ua.$$

If we choose reference points like u in the picture above in a smooth manner over some region $U \in M$, the we get a local <u>section</u> of P, a surface of reference points,



Call the section $S: U \to P$, $\pi S(x) = x$. Then we get an associated local trivialization,

$$\phi: U \times G \to P \;:(x,a) \longmapsto S(x)\,a.$$

So on a P.F.B., a local section and a local trivialization imply one another.

For the Hopf map, the choice of a reference point on a fiber F_x is the same as a <u>phase convention</u> for the normalized spinors pointing in the direction $x = (\theta, \varphi)$. And the section S is a smooth assignment of phase conventions for spinors pointing in directions (θ,φ) over some region of S^2.

Let $z_0 = \begin{pmatrix}1\\0\end{pmatrix}$ be the spinor "pointing in" the $\hat{z}$-direction, i.e., $\langle z_0 | \vec{\sigma} | z_0 \rangle = \hat{z} = \begin{pmatrix}0\\0\\1\end{pmatrix}$. Then a spinor pointing in direction (θ,φ) can be written down using spin rotation operators,

$$z(\theta,\varphi) = R_z(\varphi)\, R_y(\theta)\, z_0,$$

where the rotation about axis $\hat{a}$ by angle θ is

$$R_{\hat{a}}(\theta) = e^{-i\frac{\theta}{2}\hat{a}\cdot\vec{\sigma}} = \cos\frac{\theta}{2} - i(\hat{a}\cdot\vec{\sigma})\sin\frac{\theta}{2}.$$

So,

$$z(\theta,\varphi) = \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{+i\varphi/2} \end{pmatrix}\begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\varphi/2}\cos\theta/2 \\ e^{i\varphi/2}\sin\theta/2 \end{pmatrix}.$$

This spinor $z(\theta,\varphi)$ points in the (θ,φ) direction, but it is not continuous over the whole sphere, mainly due to the ½ angle $\varphi/2$ which gives a discontinuity on the line $\varphi=0$. We can get rid of this by multiplying by $e^{i\varphi/2}$ (a change in phase convention) to obtain the spinor

$$z_N(\theta,\varphi) = \begin{pmatrix} \cos\theta/2 \\ e^{i\varphi}\sin\theta/2 \end{pmatrix}.$$

This spinor is smooth everywhere on S^2 except at the south pole, where $e^{i\varphi}$ represents oscillations of finite amplitude that take place over a smaller and smaller region as $\theta \to \pi$. Thus the spinor is not differentiable at the south pole. The similar problem does not occur at the north pole, because $\sin\theta/2 \to 0$ there. So we call this spinor z_N to indicate that it is smooth over the northern hemisphere.