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On a P.F.B., however, G always has a natural action. In specific examples, this is usually easy to see. For example, in the frame bundle FM choose a particular frame $\{f_\mu\} \in F_x M$. Let $G = GL(m, \mathbb{R})$ act on this by

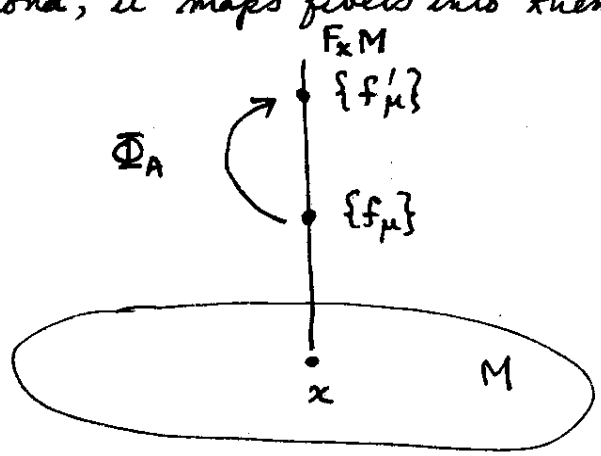
$$\Phi_A \{f_\mu\} = \{f'_\mu\}$$

$$\text{where} \quad f'_\mu = f_\nu A^\nu{}_\mu,$$

where we call the action $A \mapsto \Phi_A$, $A \in GL(m, \mathbb{R})$ and $\Phi_A : P \to P$ (here $\Phi_A : FM \to FM$). Notice that

$$\Phi_B \Phi_A = \Phi_{AB},$$

so this is a right action. Note the following things about this action. First, it did not require a local trivialization for its definition; it is global. No coordinates on M or on $F_x M$ are needed. Second, it maps fibers into themselves,



In fact, the orbit of the action is the entire fiber (the action is transitive on the fiber). Third, the action on P is free ($\Phi_A \{f_\mu\} = \{f_\mu\}$ iff $A = Id.$).

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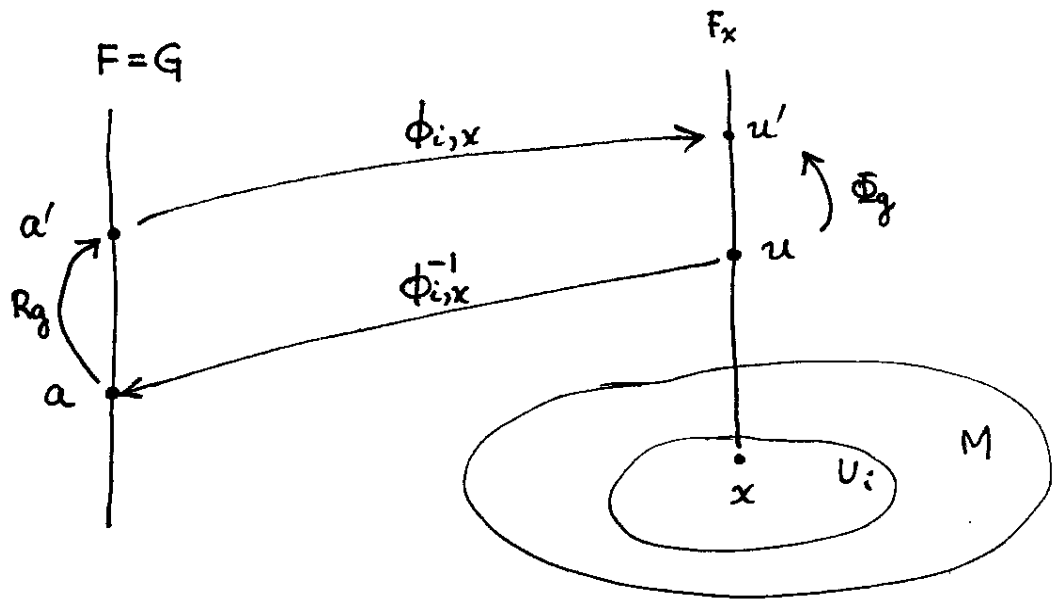
A common way of creating a P.F.B. is to allow a group G to act on a manifold. call it P. If the action is free, then the orbits of the group action are diffeomorphic to G, and we can regard them as the fibers of a bundle. We assume the action of G on P is from the right. We then define $M = P/G$, which also defines π.

The Hopf fibration is an example of such a P.F.B. It arises from letting $U(1)$ act on S^3 (as specified above). So is the construction of ~~homogeneous~~ symmetric spaces mentioned above. Here $P = G$ (not the structure group), H is a subgroup of G (H is the structure group), H acts on G by right multiplication ($g \mapsto gh$), the orbits of the action are left cosets gH, $M = G/H$ is the symmetric space, the space of left cosets, and $\pi : G \to M$. In all these cases the action of the structure group (call it G again) on P is obvious: it is just the action that was used to form the bundle.

We now define the action of G on P in the general case. Let $g \in G$. We wish to define $\Phi_g : P \to P$. Let $u \in P$. u belongs to some fiber F_x, $\pi(u) = x$, and x lies in some U_i, speaking of a set $\{(U_i, \phi_i)\}$. For $x \in U_i$, we define the action of Φ_g as in the picture,



that is,

$$u' = \Phi_g u = \phi_{i,x} R_g \phi_{i,x}^{-1} u,$$

or $$\Phi_g u = \phi_{i,x}\left(\left(\phi_{i,x}^{-1} u\right) g\right).$$

This mapping is apparently only defined over U_i, and it apparently depends on the local trivialization. But suppose $x \in U_i \cap U_j$. Then

$$\phi_{i,x} = \phi_{j,x} \circ t_{ij,x}^{-1}, \quad \text{so}$$

$$\Phi_g u = \phi_{j,x}\left[t_{ij,x}^{-1}\left(\left(t_{ij,x} \circ \phi_{j,x}^{-1} u\right) g\right)\right],$$

or by rearranging parentheses,

$$\Phi_g u = \phi_{j,x}\left(\left(\phi_{j,x}^{-1} u\right) g\right).$$

The answer does not depend on which local triv. we use, and in fact it is globally defined. Notice that Φ_g preserves fibers, the orbits <u>are</u> the fibers, the action is free on P and transitive on each

fiber, and it is a right action.

Here is another special property of a P.F.B., not shared by other fiber bundles.

<u>Thm</u>. A P.F.B. is trivial iff it possesses a global section.

<u>Proof</u>: (a). Suppose (P, M, G, π) is trivial. Then there exists $\phi: M \times G \to P$ such that $\pi\,\phi(x,g) = x$. Define $S: M \to P$ by

$$S(x) = \phi(x, e).$$

Here we can use any constant group element, e is just convenient. Then $\pi\, S(x) = \pi\,\phi(x,e) = x$, so $S(x)$ is indeed a global section.

(b) Suppose $\exists$ $S: M \to P$ such that $\pi(S(x)) = x$. Then define $\phi: M \times G \to P$ by

$$\phi(x,g) = S(x)g,$$

where we are using the right multiplication defined for any P.F.B. Then

$$\pi\,\phi(x,g) = \pi\big(S(x)g\big) = \pi(S(x)) = x,$$

where we use the fact that $S(x)$ and $S(x)g$ belong to the same fiber (right action is fiber preserving). ~~Thus, ϕ is a fiber~~ Note also that $\phi_x: G \to F_x : g \mapsto \phi(x,g) = S(x)g$ is a diffeomorphism, since the orbit of the right action is the whole fiber. (More exactly, the argument shows that ϕ_x is invertible, the fact that it is a diffeomorphism follows from the general assumption of smoothness).

Thus, ϕ is a diffeomorphism, and P (the bundle) is trivial.

As noted previously, no similar theorem holds for other types of bundles. For example, every vector bundle has a global section (the zero section) whether or not it is trivial. But here is a useful theorem regarding vector bundles.

<u>Thm.</u> A vector bundle is trivial iff the corresponding frame bundle is trivial.

We will prove this for the special case of the tangent bundle TM and the frame bundle FM (which means the bundle of frames in the tangent spaces. Every ~~frame~~ vector bundle has a corresponding frame bundle.)

(a) Suppose FM is trivial. Then by the previous theorem there exists a field of frames $\{f_\mu\}$, globally defined and smooth everywhere. Then define

$$\phi : M \times \mathbb{R}^m \to TM$$
$$: (x, (V^1, \ldots, V^m)) \longmapsto V^\mu f_\mu|_x .$$

This is a bijection hence a diffeomorphism, and fiber preserving, $\pi\,\phi(x, \cdot) = x$, so TM is trivial.

(b) Suppose TM is trivial. Then $\exists\ \phi : M \times \mathbb{R}^m \to TM$, a diffeo. such that $\pi\,\phi(x, \cdot) = x$. Let $\{E_1, \ldots, E_m\}$ be a basis in $\mathbb{R}^m$ (each E_μ is an m-vector of numbers, maybe the "unit vectors" in $\mathbb{R}^m$). Map these onto TM using ϕ, i.e.,

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define

$$e_\mu|_x = \phi(x, E_\mu).$$

Then we get a frame in each tangent space, hence a field of frames, hence a global section of FM, hence FM is trivial.

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The tangent bundle TM and frame bundle FM have the same base space M, and same structure group $GL(m,\mathbb{R})$. When we showed that these spaces actually are bundles, we used the same open cover $\{U_i\}$ of M for both these bundles, and we found that the transition functions $t_{ij}: U_i \cap U_j \to G$ were the same, that is, we found $t_{ij}(x) = J(x)$ where J is the Jacobian matrix connecting the i-coordinates with the j-coordinates on a fiber. The fibers, however, are different ($\mathbb{R}^m$ for TM, $GL(m,\mathbb{R})$ for FM).

A bundle was defined as trivial if it is possible to gauge away the transition functions, i.e., to find functions $g_i: U_i \to G$ for all i such that

$$g_i(x)^{-1}\, t_{ij,x}\, g_j(x) = e, \quad \forall\, x \in U_i \cap U_j \quad \forall\, i,j.$$

The possibility of doing this depends on the sets $\{U_i\}$ and functions t_{ij}, but not on the nature of the fiber. Thus, in the case of TM and FM, if we can gauge away t_{ij} for one bundle we can do it for the other, and TM is trivial iff FM is trivial. This is another point of view on the theorem just recently proved.

Bundles TM and FM are said to be <u>associated</u>, meaning they have the same M, G, $\{U_i\}$ and t_{ij}, but different fibers. Let us consider the problem of constructing a new bundle associated to a given (original) one, in which the fiber changes. Denote the properties of the original bundle with a 0-subscript, and the new bundle without a 0 subscript. Then we have:

original:	E_0	M	F_0	G	π_0	$\{U_i\}$	$\{\varphi_{i0}\}$	t_{ij}
new:		M	F	G		$\{U_i\}$		t_{ij}

We drop the 0-subscript on objects common to both bundles, M, G, U_i, t_{ij}, but note the fiber has changed from F_0 to F. This gives us partial information about the new bundle. Can we fill in the missing elements $(E, \pi, \{\varphi_i\})$?

Alternatively, we might image that someone has given us partial information about a bundle (the info on the 2nd line above), and asks us to reconstruct the bundle. This is the <u>reconstruction</u> problem.

We begin by constructing the locally trivial Cartesian products $U_i \times F$, and considering the disjoint union of these:

$$X = \bigcup_{\substack{i \\ \text{disjoint}}} U_i \times F .$$

This means the following. An element of X is a triplet,

$$(i, x, f), \qquad \text{where } x \in U_i, \ f \in F.$$

That is, points of X remember which U_i they came from, and

$$(i, x, f) = (j, x', f')$$
$$\text{iff} \quad i = j, \quad x = x', \quad f = f'$$

Now define a relation on X,

$$(i, x, f) \sim (j, x', f')$$

if $\quad x = x'$
$\quad f = t_{ij,x} f'$

This is an equivalence relation if $t_{ij,x}$ satisfies:

(a) $t_{ii,x} = \mathrm{id}_F \quad (\text{or } e \in G), \qquad x \in U_i$

(b) $t_{ij,x}^{-1} = t_{ji,x}, \qquad x \in U_i \cap U_j$

(c) $t_{ij,x}\, t_{jk,x} = t_{ik,x} \qquad x \in U_i \cap U_j \cap U_k.$

$t_{ij,x}$ <u>does</u> satisfy these conditions if it came from some original bundle. If not, these are extra conditions that the $t_{ij,x}$ have to satisfy in order to (re) construct the new bundle.

The equivalence relation above amounts to using glueing rules for the regions U_i that reproduce the glueing rules in the original bundle (because the t_{ij} are the same). Both bundles are given the same "twists".

Now define

$$E = \frac{X}{\sim},$$

so $\qquad$ element of $E = [(i, x, f)]$.

Then define

$$\pi : E \to M$$
$$: [(i, x, f)] = x.$$

This is meaningful, since all elements (i, x, f) of the equivalence class have the same x. The fiber over x_0 is

$$F_{x_0} = \pi^{-1}(x_0) = \{ [(i, x, f)] \mid x = x_0 \}.$$

It is diffeo. to F because $x = x_0$ is fixed, $f \in F$, and different i's are

related by the equivalence relation.

Finally, define

$$\phi_i : U_i \times F \to \pi^{-1}(U_i)$$

$$\text{or } \phi_{i,x} : F \to F_x$$
$$: f \mapsto [(i,x,f)].$$

The obvious definition. Can check that $t_{ij,x} = \phi_{i,x}^{-1}\phi_{j,x}$ (the final step), and then we have reconstructed the bundle.

The original bundle could be a vector bundle ($F_0 =$ some vector space), in which case we might choose $F = G$ to get the associated P.F.B. This will be isomorphic to the frame bundle. Or the original bundle might be a P.F.B. and new bundle a vector bundle, in which case we get vector bundles associated with the P.F.B. In this case, G usually acts on F (the vector space) by some representation. In this way we can construct the cotangent bundle and all the various tensor bundles as bundles associated with FM.

An interesting case is when one bundle is FM in GR with orthonormal frames, so that $G = L_0 \in O(1,3)$ (proper orthochronous Lorentz transformations), and we wish the associated bundle to be a spin bundle with $F = \mathbb{C}^2$ (Weyl spinors) or $\mathbb{C}^4$ (Dirac spinors). The subtlety in this case is that the structure group must be lifted to $SL(2,\mathbb{C})$ (it is not the same structure group), and as a result spin bundles do not exist over just any space-time manifold.

We now consider the behavior of bundles under maps. It turns out that bundles can be pulled back, but not generally pushed forward.

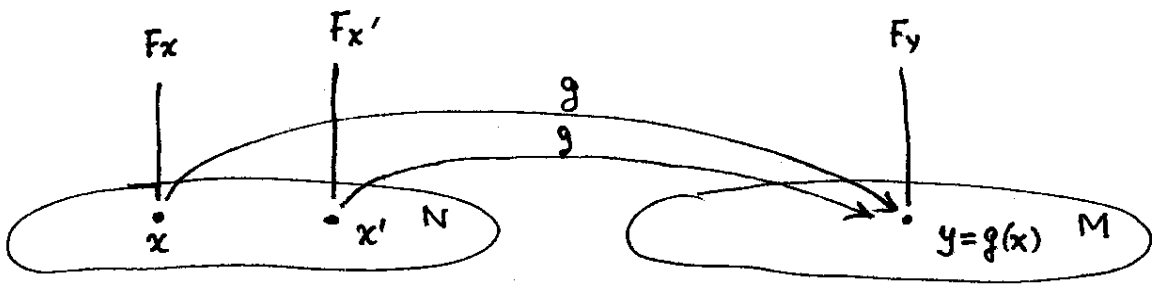
Let $E \xrightarrow{\pi} M$ represent a bundle over M with std. fiber F, and let $g: N \to M$ be a map. Here are the spaces:

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ N \arrow[r, "g"] & M \end{tikzcd}$$

It turns out that we can pull back the bundle structure over M to create a new one over N; (N will have the same std. fiber F as M).

$$\begin{tikzcd} \tilde{E} \arrow[d, "\tilde{\pi}"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "g"] & M \end{tikzcd}$$

where $\tilde{E}, \tilde{\pi}$ denote the new bundle $\tilde{E} \xrightarrow{\tilde{\pi}} N$. The idea is that fibers over M get pulled back and copied to make fibers over N.



In the picture, F_x will be made an identical copy of F_y, where $y = g(x)$. Note that g does not have to be injective. There may be more than one point of N (x, x' above) that map to a given $y \in M$. If so, both fibers F_x, $F_{x'}$ are identical copies of F_y. By pulling back M fibers to N at all points $x \in N$, we get a new bundle over N. This is the intuitive idea of the pull-back bundle.

By "identical copy" we mean that there is a natural diffeomorphism (based on the given geometrical elements) between F_x and F_y in the

diagram above (and betw. $F_{x'}$ and F_y). Of course all fibers are diffeo. to the std. fiber F and hence to each other, but not usually in a natural way. This natural diffeomorphism between F_x for any $x \in N$ and F_y for $y = g(x)$ amounts to a fiber-preserving ~~diffeo~~ map betw. $\tilde{E}$ and E, call it $\bar{g}: \tilde{E} \to E$ (the lift of $g: N \to M$), so the overall picture is

$$\begin{tikzcd} \tilde{E} \arrow[r, "\bar{g}"] \arrow[d, "\tilde{\pi}"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "g"] & M \end{tikzcd}$$

This is a commuting diagram (since $\bar{g}$ is fiber preserving).

Now to actually construct the pull-back bundle. Given data:

Orig. (over M):	E.	M	F	G	π	U_i	ϕ_i	t_{ij}
new (over N):		N	F	G				

Initially all we know is the base space N and std fiber F and structure group (the latter two assumed to be the same for both bundles).

First let us get the open cover for N. Define $V_i = g^{-1}(U_i)$. Since g is continuous (as we assume), the inverse images of open sets are open, and the V_i are open sets on N. This is one reason why pushing forward a bundle won't work in general, the forward image of an open set is not necessarily open. Moreover, the collection $\{V_i\}$ forms an open cover of N, since every $x \in N$ lies in some V_i (since $f(x) \in M$ lies in some U_i).

Next get the transition functions for the new bundle. Call them s_{ij}. Then we have

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$$t_{ij} : U_i \cap U_j \to G$$

$$s_{ij} : V_i \cap V_j \to G.$$

Noting that $g^{-1}(U_i \cap U_j) = V_i \cap V_j$, it is logical to define s_{ij} as the pull back of t_{ij},

$$s_{ij} = g^* t_{ij},$$

i.e. $$s_{ij,x} = s_{ij}(x) = t_{ij}(g(x)) = t_{ij,g(x)}.$$

We now have everything needed $(N, F, G, \{V_i\}, s_{ij})$ to proceed with the reconstruction program, giving us $\tilde{E}, \tilde{\pi}$, and ψ_i (the local trivializations).

Following the reconstruction program above, with changes of notation, we have

$$X = \bigcup_{\substack{i \\ \text{disjoint}}} V_i \times F$$

$(i, x, f) \in X$ where $x \in V_i$, $f \in F$

$(i, x, f) \sim (j, x', f')$ if $x = x'$, $f = s_{ij,x} f'$.

Note that the $s_{ij,x}$ satisfy the consistency requirements (a)(b)(c) since the $t_{ij,y}$ do. Then

$$\tilde{E} = \frac{X}{\sim},$$

$$[(i, x, f)] \in \tilde{E},$$

$$\tilde{\pi} : \tilde{E} \to N : [(i, x, f)] \mapsto x.$$

$$\psi_{i,x} : F \to F_x : f \mapsto [(i, x, f)].$$

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So we have reconstructed the pull-back bundle. Denote the original bundle (over M) by E for short, and the pull-back bundle by g^*E.

Finally, to define the lift $\bar{g}: \tilde{E} \to E$, define

$$\bar{g} : [(i, x, f)] = \phi_{i, g(x)} f.$$

This maps F_x diffeomorphically onto F_y, where $y = g(x)$. Can check that this defn of $\bar{g}$ is independent of the representative element of the equivalence class.

Now consider the same basic picture, but with two maps $f, g : N \to M$:

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ N \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & M \end{tikzcd}$$

So we get 3 bundles, E over M and f^*E, g^*E over N. Then we have a theorem:

<u>Thm</u>: If f is homotopic to g, then f^*E is equivalent (i.e. compatible with) to g^*E.

No proof here. Intuitively, this says that if f^*E continuously changes into g^*E, then the topology of the bundle can't change either.

Another variation on the above. Set $N = M$, then

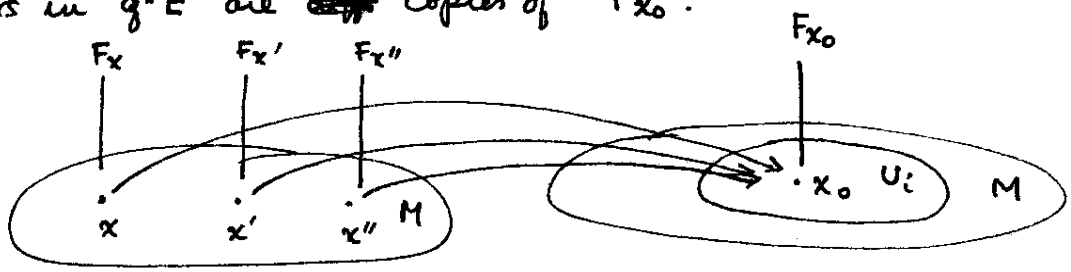
$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ M \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & M \end{tikzcd}$$

gives us 3 bundles over M, E, f^*E, g^*E. Now let

$$f: M \to M: x \mapsto x, \qquad f = id_M$$

$$g: M \to M: x \mapsto x_0 \qquad \text{const. map.}$$

Since $f = id_M$, $f^*E = E$. As for g, since it is a constant map, all fibers in g^*E are ~~diff~~ copies of F_{x_0}:



So there is a natural isomorphism of each fiber ~~to~~ of g^*E with one fiber F_{x_0}, hence there exists a global trivialization of g^*E, hence g^*E is trivial. Another way to see the same thing is to note that if $x_0 \in U_i$, then $g^{-1}(U_i) = M$. Thus, $V_i = M$, and M is covered (for the purpose of bundle g^*E) by a single chart. Again, this implies a trivial bundle (the local triv. on this chart is a global triv.)

Now suppose M is <u>contractible</u>, so $f \sim g$. Then by the previous thm we have a new thm: (useful and important.)

$\boxed{\text{Thm. If M is contractible, then any bundle over M is trivial.}}$

Now consider how bundles get used in quantum mechanics and field theory. Let ψ be the wave function for a nonrelativistic charged spinless particle. We usually think of ψ as a complex-valued scalar field:

$$\psi: \mathbb{R}^3 \to \mathbb{C}.$$

Notice that $|\psi|^2$ is physically measurable at a point $\vec{x}$ of space, since it