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be assigned smoothly over the sphere.

In the above examples we constructed a fiber bundle by starting with the base space and adding fibers at each point. In the next example we start with E and work down to M .

Let $E = S^3$. We visualize $S^3 \subset \mathbb{R}^4$ as the set of unit vectors in $\mathbb{R}^4 \cong \mathbb{C}^2$. Let (x_1, x_2, x_3, x_4) be coordinates on \mathbb{R}^4 , let

$$z_1 = x_1 + ix_2$$

$$z_2 = x_3 + ix_4$$

Let

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2,$$

so that

$$|z|^2 = |z_1|^2 + |z_2|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

Think of z as a 2-component spinor. Thus, the points of S^3 can be thought of as normalized spinors. In the following assume

$$|z|^2 = 1, \text{ so } z \in S^3.$$

We define an action of $U(1)$ on S^3 by

$$z \mapsto e^{i\alpha} z, \quad e^{i\alpha} \in U(1), \quad 0 \leq \alpha < 2\pi.$$

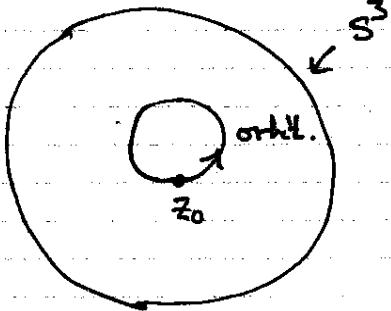
We are changing the "overall phase" of the spinor. This action is free, which means that the isotropy subgroup is trivial (cf (that is, if $z = e^{i\alpha} z$, then $\alpha = 0$)). Thus the orbit of the $U(1)$ action is a circle (diffeo. to $U(1)$), and we have a foliation of S^3 into a 2D family of circles.

Now we define

$$M = \frac{S^3}{U(1)}$$

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that is, M is the quotient space of this group action. The projection map $\pi: S^3 \rightarrow M$ is defined by $\pi(z) = \pi(z') \Leftrightarrow z' = e^{i\alpha}z$, that is, π maps an entire orbit in S^3 onto a point of M . Thus, the orbit is the fiber F_x over $x \in M$; x is a label of the orbit. The standard fiber is $F = U(1) \cong S^1$. This is another example of a principle fiber bundle.

What is the space M ? Suppose we wished to find coordinates on M . These would be scalar fields on M , say, $f: M \rightarrow \mathbb{R}$. Such scalar fields could be lifted to S^3 by the pullback, $\pi^*f: S^3 \rightarrow \mathbb{R}$, such scalars π^*f have the property that they are constant on the orbits of the $U(1)$ action. Conversely, any scalar $: S^3 \rightarrow \mathbb{R}$ const. on the $U(1)$ orbits can be projected to a scalar $: M \rightarrow \mathbb{R}$ using π .

A scalar on S^3 that is const. on the $U(1)$ orbits must have the same value at any two points z and $z' = e^{i\alpha}z$. An obvious way to create such scalars is to use bilinear quantities, $(z^\dagger A z) = \langle z | A | z \rangle$ in spinor language, where A is a 2×2 matrix.

Let us define $H: S^3 \rightarrow \mathbb{R}^3$ by

$$H(z) = \langle z | \vec{\sigma} | z \rangle = \vec{n}(z),$$

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where $\vec{\sigma}$ are the Pauli matrices, and $\vec{n}(z)$ is notation for the value of the function. $\vec{n}(z)$ is a real vector because $\vec{\sigma}$ is Hermitian, and of course $\vec{n}(z)$ is constant on $U(1)$ orbits.

Then we have two simple theorems:

(a) $\vec{n}(z)$ is a unit vector, $|\vec{n}(z)|=1$. So write $\hat{n}(z)$ instead, and regard the map H as

$$H: S^3 \rightarrow S^2$$

where $\hat{n}(z)$ indicates a point on $S^2 \subset \mathbb{R}^3$.

(b) $\hat{n}(z) = \hat{n}(z')$ iff $z' = e^{i\theta} z$. This part shows that orbits in S^3 are placed in one-to-one correspondence with points of S^2 . Thus, $M = S^3$, and we have

$$\frac{S^3}{U(1)} = S^2,$$

This is called the Hopf fibration, and H is the Hopf map. It gives us an example of a circle bundle ($U(1) \cong S^1$) over S^2 .

Another example. Let G = any Lie group, and H = a Lie subgroup. Let G/H = the space of (left, say) cosets. Then we have the structure of a (principal) fiber bundle, in which $E = G$, $F = H$, $M = G/H$, $\pi: G \rightarrow G/H: g \mapsto [g] = gH$.

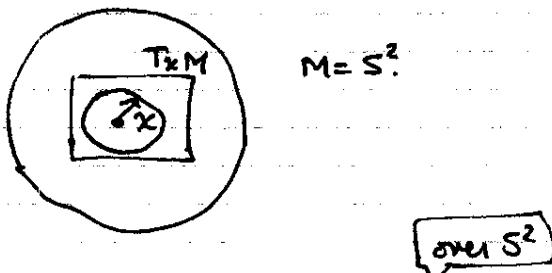
In an early homework it was shown that

$$\frac{SO(3)}{SO(2)} = S^2,$$

where $SO(2)$ is the subgroup of rotations about the z -axis (say).

But $SO(2) \cong S^1$ and $SO(3) \cong \mathbb{RP}^3$, so we have again 11/20/08
 an example of a circle bundle over S^2 , but this time the entire
 space $E = \mathbb{RP}^3$ (not S^3 as in the Hopf fibration). The difference
 is in the twistings applied to the way circles are attached to S^2 .

Another circle bundle over S^2 is easy to construct. Give S^2 the
 standard metric by embedding in \mathbb{R}^3 , and consider the unit circle
 bundle, i.e., the set of vectors ~~in~~ x such that $|x|=1$:



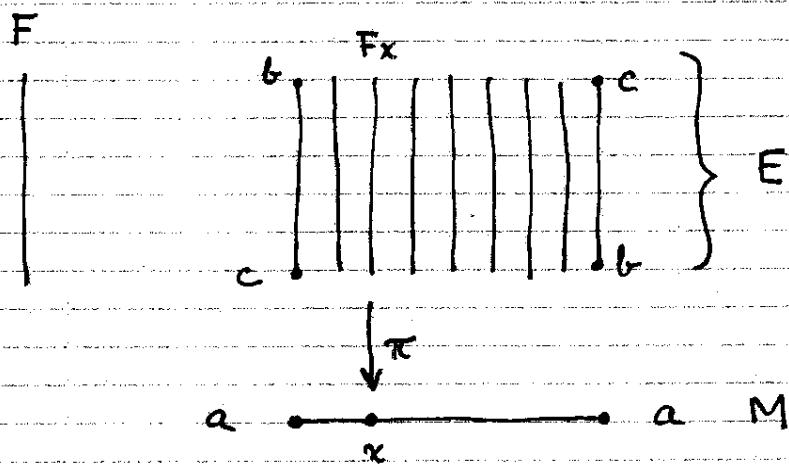
It is an exercise to see if this circle bundle is different from the two previous ones.

Two things to note about these examples. First, note that in general there is no natural identification between the standard fiber and the fibers F_x over points $x \in M$. For example, in the tangent bundle TM , each $T_x M$ is certainly diffeo. to \mathbb{R}^n , but we get a specific identification of these two spaces only when we introduce a basis in $T_x M$. Similarly, in the frame bundle FM we obtain an identification of the fiber with $GL(n, \mathbb{R})$ only after a specific (reference) frame is chosen; then all other frames are related to this one by group actions. To say this another way, in the case of a principle fiber bundle (such as the frame bundle) each fiber is diffeo. to the structure group G , but unlike the group G itself, the fiber has no preferred "identity element."

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M should second, note that the base space E , not properly be considered a submanifold of E , but rather a quotient space. This is clear in examples above such as the Hopf fibration, but it true also in other examples such as the Möbius strip. In fact, the drawings above of the Möbius strip as a fiber bundle were wrong, we should instead have drawn something like this:



In fact, it's easy to show that M can be identified with a submanifold of a bundle only when a global section exists (something we don't want to assume in general).

Now we turn to the official definition of a fiber bundle.

A coordinate bundle consists of the following.

1. Spaces:

(a) M the base space

(b) F the standard fiber

(c) E the entire space (or "the bundle")

+ more later

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2. A surjective map, $\pi: E \rightarrow M$ (smooth, of course).

We define $F_x = \pi^{-1}(x)$ (for $x \in M$) as the fiber over x .

F_x is required to be diffeomorphic to the standard fiber F .

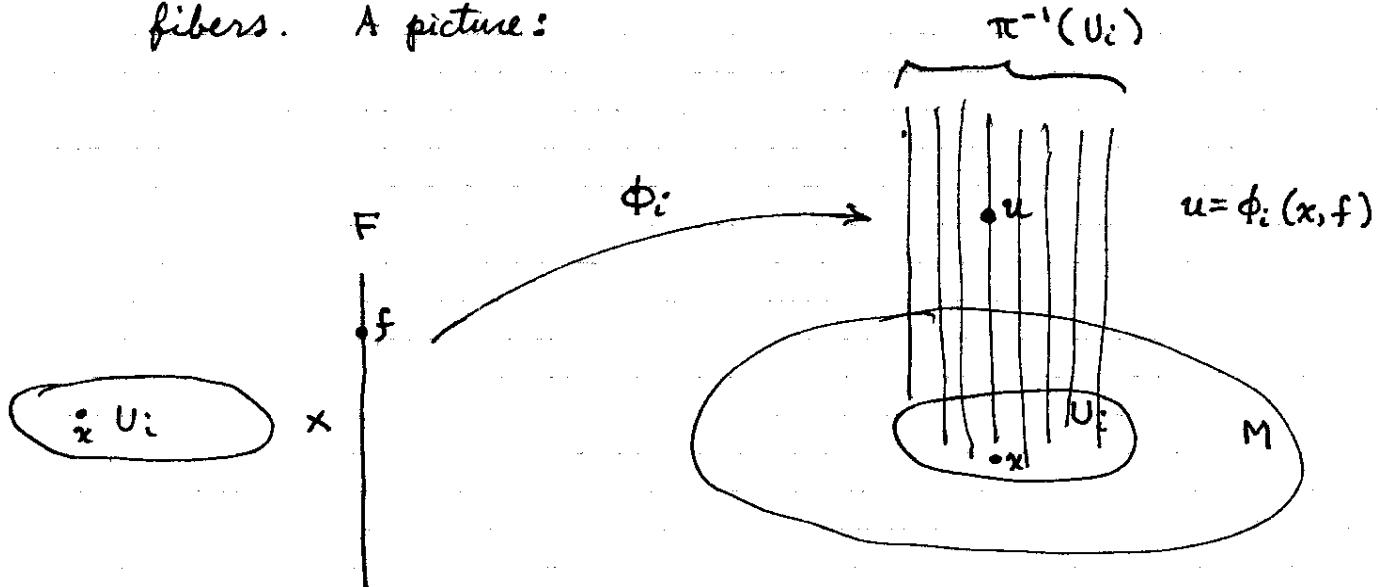
3. A set $\{(U_i, \phi_i)\}$, where $\{U_i\}$ is an open cover of M , and the ϕ_i are diffeomorphisms,

$$\phi_i: U_i \times F \rightarrow \pi^{-1}(U_i)$$

such that $\pi \phi_i(x, f) = x$ for $\forall x \in U_i$, $f \in F$. The

ϕ_i are called local trivializations. These maps make precise the notion that a fiber bundle is locally a Cartesian product.

The condition $\pi \phi_i(x, f) = x$ means that the maps preserve fibers. A picture:

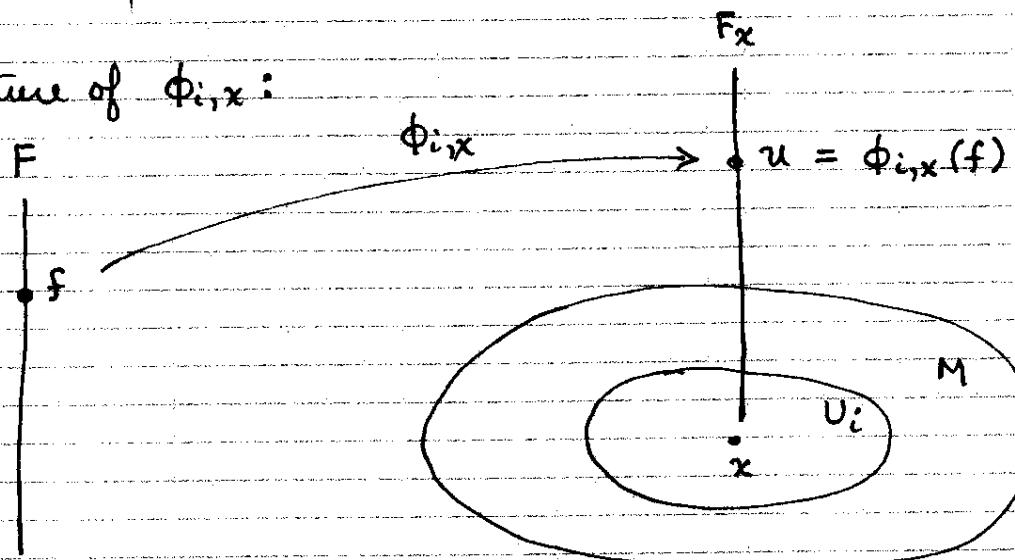


It is useful to define ϕ_i restricted to one fiber (the one over x), by

$$\phi_{i,x}: F \rightarrow F_x,$$

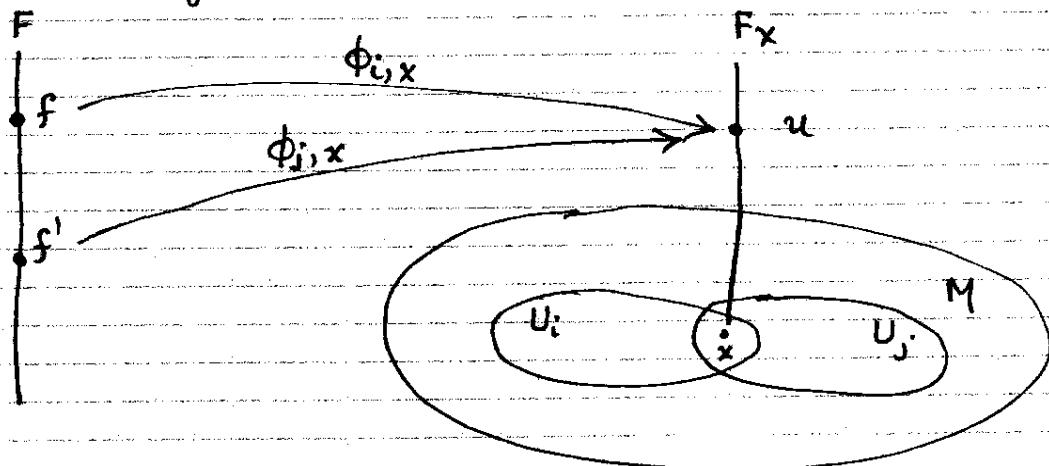
$$\phi_{i,x}(f) = \phi_i(x, f).$$

A picture of $\phi_{i,x}$:



We can think of $\phi_{i,x}$ as putting coordinates on the fiber F_x (labelling points on F_x by points f on the standard fiber.)

In an overlap region there will be two "coordinate systems",



The "coordinate transformation" is a map: $F \rightarrow F$, taking f' to f in the picture above. It is denoted $t_{ij,x}: F \rightarrow F$,

$$t_{ij,x}: F \rightarrow F \quad (x \in U_i \cap U_j)$$

$$t_{ij,x} = \phi_{i,x}^{-1} \circ \phi_{j,x} \quad (\text{defn of } t_{ij,x}).$$

The maps $t_{ij,x}$ are diffeomorphisms: $F \rightarrow F$, and so belong to the ∞ -dimensional group $\text{Diff}^F(M)$. But in practice, the $t_{ij,x}$ usually belong to a much smaller subgroup of $\text{Diff}(M)$, which we will call G . For example, in the tangent bundle, the standard fiber F is a vector space, and the $t_{ij,x}$ usually consist of linear transformations, so $G = GL(m, \mathbb{R})$ or perhaps a subgroup of $GL(m, \mathbb{R})$.

Thus we add one more space to the list under 1. above:

1. (d). ~~To~~ G the structure group.

It would be possible to describe G as a subgroup of $\text{Diff}(F)$, but sometimes we want to use ~~the~~ the same G for different kinds of fibers, so we prefer to think of G as an abstract group with an action on F . ~~This is to modify our~~ We require this action to be effective because that means that G is isomorphic to a subgroup of $\text{Diff}(F)$. So we modify the above by saying,

4. The structure group G has an effective action on F .

If $g \in G$ and $f \in F$, we write simply gf instead of $\mathfrak{E}af$ or some more complicated notation. ~~so~~ G acts on F from the left. The maps $t_{ij,x}: F \rightarrow F$, which ~~change~~ map "j-coordinates" on F_x into "i-coordinates", to coincide with the action of some element in G ; thus we may equally well think of $t_{ij,x} \in G$.

We require G to act from the left because that's what the maps

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$t_{ij,x}$ do, according to the definition given above. But this is just a convention (the way the $t_{ij,x}$ were defined). There are various dictates in a standard presentation of fiber bundle theory about the direction (left, right) in which a group acts, but you will find in actual applications that you may want to reverse these.

The maps $t_{ij,x} \in G$ can also be written in another notation,

$$t_{ij} : U_i \cap U_j \rightarrow G : x \mapsto t_{ij,x}.$$

The maps t_{ij} or $t_{ij,x}$ are called transition functions. They specify the coordinate transformations on the fibers F_x in overlap regions. This is taking the point of view that the local trivializations provide coordinates on the fibers over the sets U_i (by labelling points of those fibers with points on the standard fiber). This is often a useful point of view.

We note that the transition functions satisfy the following properties:

$$(a) \quad t_{ii,x} = \text{id}_F \text{ (or } e \in G\text{)}, \quad x \in U_i.$$

$$(b) \quad t_{ij,x}^{-1} = t_{ji,x} \quad x \in U_i \cap U_j.$$

$$(c) \quad t_{ij,x} t_{jk,x} = t_{ik,x} \quad x \in U_i \cap U_j \cap U_k.$$

These follow immediately from the definition, $t_{ij,x} = \phi_{i,x}^{-1} \phi_{j,x}$, and are not additional requirements. We use these later.

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The above is the definition of a coordinate bundle. It obviously has a lot of arbitrariness in it, due to the fact that there are huge numbers of possible open covers $\{U_i\}$, and likewise for the maps ϕ_i . Suppose we have another coordinate bundle with the same E, M, F, G, π but set $\{(V_i, \psi_i)\}$ in place of $\{(U_i, \phi_i)\}$. The new coordinate bundle, with $\{(V_i, \psi_i)\}$, satisfies all the requirements above of a coordinate bundle, in particular, the new transition functions $\psi_{i,x}^{-1} \phi_{j,x} \in G$. We will regard the new coord. bundle as compatible or equivalent to the previous bundle if the transition functions connecting the new and old open sets belong to the structure group,

$$\psi_{i,x}^{-1} \phi_{j,x} \in G, \quad \forall x \in V_i \cap U_j; \\ \forall i, j.$$

Equivalently, we can throw the sets $\{(U_i, \phi_i)\}, \{(V_i, \psi_i)\}$ together (take their union), and require that the new set gives a coordinate bundle. *

This compatibility condition is an equivalence relation, so the space of all coordinate bundles with given E, M, F, G, π but different $\{(U_i, \phi_i)\}$ breaks up into equivalence classes. We call one such equivalence class a fiber bundle.

This definition of a fiber bundle obviously has a lot in common with the definition of a differentiable structure on a manifold (an equivalence class of atlases, each of which involves open ~~ssets~~ covers $\{U_i\}$ and maps $\{\phi_i\}$ etc.).

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of some subset of the equivalence class of coordinate bundles that make up a fiber bundle involves transition functions that belong strictly to a subgroup H of the structure group G , then it may be desirable to redefine the structure group as H and to redefine the fiber bundle as the given subset of the original equivalence class (which becomes a new equivalence class w.r.t. H). This is the reduction of the structure group. For example, normally we think of the structure group of the tangent bundle TM as $GL(m, \mathbb{R})$, but if we restrict consideration to orthonormal frames (on a Riemannian manifold) then we may wish to consider the structure group to be $O(m)$. For another example, most books say the structure group of the Möbius strip is \mathbb{Z}_2 , because that is the smallest group necessary to get the essential twist. But you could consider the structure group to be a larger group (such as $GL(1, \mathbb{R})$, if you take the fiber to be $F = \mathbb{R}$).

Suppose two coordinate bundles in the equivalence class making up a fiber bundle have the same open cover $\{U_i\}$ of M , but different local trivializations, say, $\{\tilde{\Phi}_i\}$ and $\{\tilde{\Phi}'_i\}$. Then the compatibility condition ~~says~~ says,

$$\tilde{\Phi}_{i,x}^{-1} \tilde{\Phi}'_{j,x} \in G.$$

In particular, setting $i=j$, we get a G -valued function over U_i ,

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$$\tilde{\phi}_{i,x}^{-1} \tilde{\phi}_{i,x} \equiv g_i(x) \in G,$$

$g_i: U_i \rightarrow G$. Equivalently,

$$\tilde{\phi}_{i,x} = \phi_{i,x} g_i(x).$$

We may say, we apply $g_i(x)$ to permute the points of the standard fiber F before applying $\phi_{i,x}$ to get coordinates (the new, \sim coordinates) on actual fiber F_x . This transformation (from ϕ to $\tilde{\phi}$) can be regarded as a gauge transformation. A coordinate bundle with a specific $\{U_i, \phi_i\}$ is equivalent to other coordinate bundles with the same $\{U_i\}$ but with new ϕ_i related by gauge transformations to the old one. The set of gauge transformations is quite large (the functions $g_i: U_i \rightarrow G$ need only be smooth), so there are a lot of ^{equivalent} coordinate bundles with the same ~~$\{U_i\}$~~ $\{U_i\}$. And of course there are a lot of ways of choosing an open cover $\{U_i\}$ (although it must be sufficiently fine that local trivializations exist). This is a way of seeing that the equivalence class of coordinate bundles making up a fiber bundle is very large.

~~Suppose for a topological coordinate bundle it is possible to choose~~

Under a gauge transformation, the transition functions transform according to

$$\tilde{t}_{ij,x} = \tilde{\phi}_{i,x}^{-1} \tilde{\phi}_{j,x} = g_i(x)^{-1} t_{ij,x} g_j(x).$$

They are still elements of G , as they must be.

Suppose for a given coordinate bundle it is possible to choose $g_i: U_i \rightarrow G$ and $g_j: U_j \rightarrow G$ (for given i, j) such that

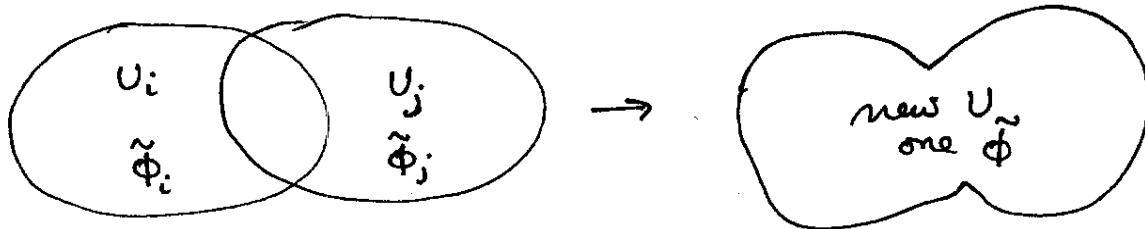
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$$\tilde{t}_{ij,x} = g_i(x)^{-1} t_{ij,x} g_j(x) = e,$$

$$\forall x \in U_i \cap U_j, \text{ i.e., } t_{ij,x} = g_i(x)g_j(x)^{-1}.$$

Then $\tilde{\Phi}_{i,x} = \tilde{\Phi}_{j,x}$ on $U_i \cap U_j$, and we have "gauged away" the transition fns on the overlap region. Then we might as well combine regions U_i and U_j (take their union) and replace $\tilde{\Phi}_i, \tilde{\Phi}_j$ by a new $\tilde{\Phi}$ defined on the union:



If, for a given coordinate bundle, it is possible to gauge away all the transition functions in all the overlap regions, i.e., if it is possible to find gauge functions $g_i: U_i \rightarrow G$ such that $t_{ij,x} = g_i(x)g_j(x)^{-1}$, $\forall x \in U_i \cap U_j$, $\forall i, j$, then we say that the fiber bundle (with this coord. bundle in its equiv. class) is trivial. We may take this as the official definition of triviality.

In view of the remarks above, in the case of a trivial fiber bundle we can merge all the U_i together into a single U , which is their union. But this is M itself, since $\{U_i\}$ is a cover of M . As for the corresponding local trivialization, it is now a global trivialization, $\Phi: M \times F \rightarrow \pi^{-1}(M) = E$, that preserves fibers, $\pi \Phi(x, f) = x$. The converse of this is true, too. Thus, a fiber

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is trivial iff it contains (its equivalence class contains) a coordinate bundle with just one $U = M$, and ~~one~~ one corresponding local (= global) trivialization, a fiber-preserving diffeomorphism

$$\phi: M \times F \rightarrow E,$$

$$\pi \circ \phi(x, f) = x.$$

Thus we connect the official definition of triviality with the less formal remarks made previously.

Note that if you have just one (U, ϕ) (for a trivial bundle), then there are no transition functions, and you might as well ~~reduce G to the trivial group {e}~~. You don't have to do this, but you can.

Now let's look at some examples of fiber bundles (previously discussed) and show that they actually are fiber bundles according to the official definition. Start with the tangent bundle. As for the spaces, we have $E = TM = \bigcup_{x \in M} T_x M$, $M = \text{same } M$ in both contexts, $F = \mathbb{R}^m$, $G = GL(m, \mathbb{R})$. Projection $\pi: TM \rightarrow M$ defined by $\pi(T_x M) = x$, so $\pi^{-1}(x) = T_x M = F_x$ (fiber over x). To complete the official defn, we need $\{\psi_i, \phi_i\}$. As for the open cover $\{U_i\}$, get this from an atlas on M . The atlas puts coordinates on the open sets U_i ; let coordinates on two of them (i and j) be

$$x^\mu \text{ coords on } U_i, \quad e_\mu = \frac{\partial}{\partial x^\mu},$$

$$x'^\mu \text{ coords on } U_j, \quad e'_\mu = \frac{\partial}{\partial x'^\mu}.$$

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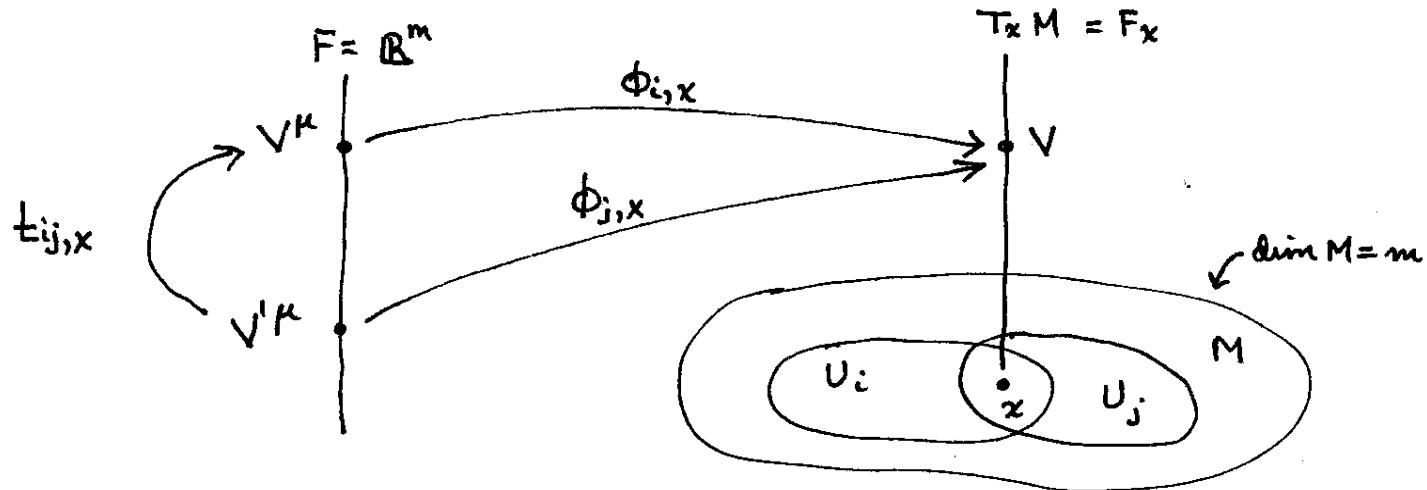
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Then we let the local trivializations be given by

$$\phi_{i,x} : \mathbb{R}^m \rightarrow T_x M : (v^1, \dots, v^m) \mapsto v^\mu e_\mu|_x, \quad (x \in U_i)$$

$$\phi_{j,x} : \mathbb{R}^m \rightarrow T_x M : (v'^1, \dots, v'^m) \mapsto v'^\mu e'_\mu|_x \quad (x \in U_j).$$

Now let $x \in U_i \cap U_j$, and think of v^μ and v'^μ as two sets of coordinates for one vector $v \in T_x M$. The map $t_{ij,x}$ maps us from the j -coordinates to the i -coordinates (v'^μ to v^μ):



By chain rule,

$$e'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} e_\nu,$$

or

$$e'_\mu = e_\nu J^\nu_\mu$$

where

$$J^\nu_\mu = \frac{\partial x^\nu}{\partial x'^\mu} = \text{Jacobian}, \det J \neq 0 \text{ so } J \in GL(m, \mathbb{R}).$$

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Thus we find

$$v^\mu = J^\mu_{\nu} v^\nu,$$

the coordinates are related just by matrix multiplication (the obvious action of $GL(m, \mathbb{R})$ on \mathbb{R}^m). This is a left action, as required, and

$$t_{ij,x} = J(x) \in G, \quad x \in U_i \cap U_j.$$

Thus TM is a fiber bundle according to the official definition.

As an exercise you may show that if you use an equivalent (but compatible) atlas, the resulting coordinate bundle is equivalent to the old one.

Next we consider the frame bundle. This is an example of a principal fiber bundle, so we define that first.

A principal fiber bundle is one for which the fiber is the same as the structure group, $F=G$. We also require the ^{left} action of G on $F=G$ (required for the transition functions) be left multiplication. We write P instead of E for a principal fiber bundle. Since $F=G$, the list of spaces is (P, M, G) .

For the frame bundle we let $F_x M$ be the set of all frames in $T_x M$, we define

$$FM = \bigcup_{x \in M} F_x M,$$

$$P = FM$$

and we define $\pi: FM \rightarrow M$ by $\pi(F_x M) = x$. We set ~~$P = FM$~~ , $G = GL(m, \mathbb{R})$, and use the same π for the bundle, so that $\pi^{-1}(x) = F_x M = F_x$ (the fiber over x is the frame space over x).

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To create the required $\{(U_i, \phi_i)\}$ we use an atlas to define the $\{U_i\}$ and put coordinates x^μ and x'^μ , and frames $e^\mu = \partial/\partial x^\mu$ and $e'_\mu = \partial/\partial x'^\mu$, on open sets U_i and U_j , respectively, as above with the tangent bundle.

The maps ϕ_i, ϕ_j are defined by

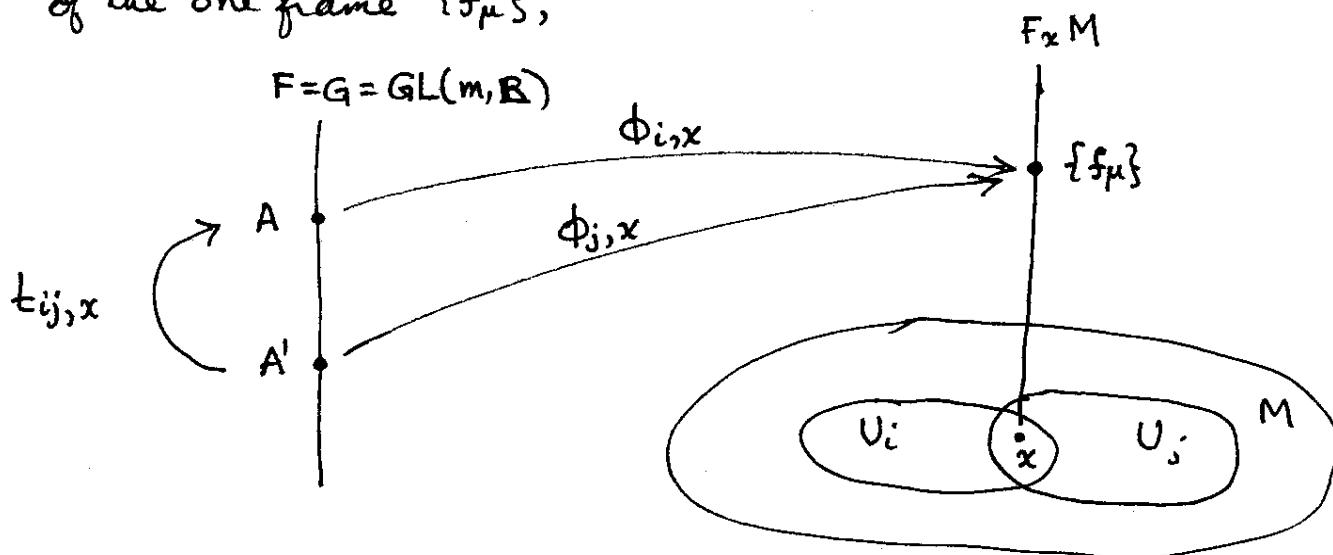
$$\Phi_{i,x} : GL(m, \mathbb{R}) \rightarrow F_x M : A \mapsto \{f_\mu\}, \quad (x \in U_i)$$

$$\text{where } f_\mu = e_\nu A^\nu{}_\mu$$

$$\text{and } \Phi_{j,x} : GL(m, \mathbb{R}) \rightarrow F_x M : A' \mapsto \{f'_\mu\}, \quad (x \in U_j)$$

$$\text{where } f'_\mu = e'_\nu A'^\nu{}_\mu$$

Here A, A' are matrices $\in GL(m, \mathbb{R})$. If we assume $x \in U_i \cap U_j$ and set $f_\mu = f'_\mu$ in the above, so that A, A' are the two coordinates of the one frame $\{f_\mu\}$,



then

$$f_\mu = e_\nu A^\nu{}_\mu = e'_\nu A'^\nu{}_\mu = e_\sigma J^\sigma{}_\nu A'^\nu{}_\mu$$

where J is the Jacobian as before. In matrix language

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this is

$$A = JA'$$

~~Abel's~~

or

$$t_{ij,x} = J(x) \in GL(m, \mathbb{R}).$$

So the frame bundle
is a P.F.B.

These are the same transition functions as in the tangent bundle.

They act on $F=G$ by left multiplication, as required of a P.F.B.

The space of fiber bundles is divided into the principal fiber bundles and everything else. Vector bundles are included in the everything else. Principal fiber bundles have properties not shared by other bundles. In particular the structure group has an action on P in the case of a P.F.B. We will require this action to be a right action, in order to conform with the conventions established above. In a particular application, you may find it more convenient to think of the action of G on P as a left action, in which case you should revise all the definitions above so that $t_{ij,x}$ are right actions.

For a general fiber bundle (not principal) G in general has no natural action on E . For example, in the tangent bundle $G=GL(m, \mathbb{R})$ and $T_x M$ is a vector space, but we cannot specify an action of G on $T_x M$ until a basis is chosen in $T_x M$. In fact, such bases are selected when we set up the local trivializations, but these are only defined over the U_i , they don't usually agree in the overlaps, and they depend on the choice of the U_i and ϕ_i . Thus there is no natural action of G on $E=TM$.