

$$\int d^m x \sqrt{|g|} \alpha^\mu_{;\mu} = \int d^m x (\sqrt{|g|} \alpha^\mu)_{,\mu} = 0.$$

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You have to convert a covariant deriv. to an ordinary deriv. if you want to integrate by parts.

Now Hodge \* and Maxwell eqns (EM). Already noted,

$$S_{EM} = \langle F, F \rangle = \int F_A * F.$$

~~Maxwell~~ Maxwell eqns in SR:

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \quad \text{or } F = dA, \quad F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$$

$$F_{[\mu\nu;\sigma]} = 0 \quad \text{or } dF = 0$$

$$F^{\mu\nu}_{,\nu} = J^\mu$$

$$J^\mu_{,\mu} = 0$$

We use the comma goes to semicolon rule to put these into GR. For example,

$$F_{[\mu\nu;\sigma]} = 0.$$

Question: does this still mean  $dF = 0$ ? (There are extra terms involving  $\Gamma$  from the covariant derivatives). Answer: Yes, because in the LC connection, all the  $\Gamma$  terms cancel when computing the components of an exterior derivative of any ~~Krakatoo-valued~~ (real-valued, i.e., not Lie algebra-valued) form. Thus,

$$F_{[\mu\nu;\sigma]} = 0 \Rightarrow F_{[\mu\nu;\sigma]} = 0 \Rightarrow dF \stackrel{?}{=} 0 \Rightarrow$$

$$\Rightarrow F = dA \quad (\text{Poincaré lemma}) \Rightarrow F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$$

$$\Rightarrow F = A_{\nu;\mu} - A_{\mu;\nu}.$$

charge conservation

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As for  $J^\mu_{;\mu} = 0$  (in SR) it becomes  $J^\mu_{;\mu} = 0$  (in GR)

Now define

$$J = J_\mu dx^\mu \quad (\text{current 1-form}),$$

and charge conservation becomes

$$d^+ J = 0.$$

Finally, as for  $F^{\mu\nu}_{;\nu} = J^\mu$  (in SR), it becomes  $F^{\mu\nu}_{;\nu} = J^\mu$  (in GR). It can be shown (exercise for you) that this is equivalent to

$$d^+ F = J,$$

which is consistent with charge conservation because  $d^+ J = d^+ d^+ F = 0$ .

Summary of Maxwell eqns:

$F = dA, \quad d^+ J = 0.$
$dF = 0$
$d^+ F = J$

We can use these eqns to get a wave eqn. for  $A$ :

$$d^+ F = d^+ dA = J$$

Although  $d^+ d$  acting on scalars (we saw above) is (minus) the covariant Laplacian (i.e., d'Alembertian in space-time), this is not quite true for forms of higher rank ( $r \geq 1$ ). For ab. forms we define,

$$\Delta = d^+ d + dd^+.$$

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this agrees with the case of scalars since  $d^+ f = 0$  (any scalar  $f$ ), 11/18/08

so

$$\Delta f = d^+ df.$$

But on a 1-form such as  $A$  we have

$$\Delta A = J - dd^+ A.$$

The term on the RHS vanishes if we choose Lorentz gauge,  $d^+ A = 0$ .

[Think:  $-\nabla^2 \vec{A} = \vec{J} - \nabla(\nabla \cdot \vec{A})$  in NR magnetostatics.]

Now we explore the properties of the operator  $\Delta$ .

$$\boxed{\begin{aligned}\Delta &= d^+ d + d d^+ \\ &= (d + d^+)^2\end{aligned}} \quad (\text{defn})$$

, since  $d^2 = d^{+2} = 0$ .

Actually, to simplify the functional analysis it helps to assume that  $M$  is also compact.

In the following we assume the Riemannian case, so  $g$  is positive def.

This means that  $\langle , \rangle$  is also positive def., so that

$$\langle \alpha, \alpha \rangle \geq 0, \text{ and } \langle \alpha, \alpha \rangle = 0 \text{ iff } \alpha = 0. \quad (\text{any form}).$$

First note that  $\Delta$  is ~~not~~ Hermitian,

$$\begin{aligned}\langle \alpha, \Delta \beta \rangle &= \langle \alpha, d^+ d \beta \rangle + \langle \alpha, d d^+ \beta \rangle \\ &= \langle d \alpha, d \beta \rangle + \langle d^+ \alpha, d^+ \beta \rangle \\ &= \langle d^+ d \alpha, \beta \rangle + \langle d d^+ \alpha, \beta \rangle \\ &= \langle \Delta \alpha, \beta \rangle.\end{aligned}$$

(More simply, just use the rules of  $t$  on products of operators, as in QM).

Next note that  $\Delta$  is a positive definite nonnegative definite operator,

i.e.,  $\langle \alpha, \Delta \alpha \rangle \geq 0 \quad \forall \alpha$ . Proof is easy:

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$$\begin{aligned}\langle \alpha, \Delta \alpha \rangle &= \langle \alpha, d^+ d\alpha \rangle + \langle \alpha, dd^+ \alpha \rangle \\ &= \langle d\alpha, d\alpha \rangle + \langle d^+ \alpha, d^+ \alpha \rangle \geq 0\end{aligned}$$

since both terms are ~~pos~~  $\geq 0$ . In fact, because  $\langle , \rangle$  is pos. def., we have more:

$$\langle \alpha, \Delta \alpha \rangle = 0 \quad \text{iff} \quad d\alpha = 0 \text{ and } d^+ \alpha = 0.$$

In fact, there is more than this. If  $d\alpha = 0$  and  $d^+ \alpha = 0$ , then  $d^+ d\alpha = 0$  and  $dd^+ \alpha = 0$ , so  $\Delta \alpha = 0$ . But  $\Delta \alpha = 0 \Rightarrow \langle \alpha, \Delta \alpha \rangle = 0 \Rightarrow d\alpha = 0$  and  $d^+ \alpha = 0$ . So altogether we have

$$\langle \alpha, \Delta \alpha \rangle = 0 \iff \left( \begin{array}{l} d\alpha = 0 \text{ and} \\ d^+ \alpha = 0 \end{array} \right) \iff \Delta \alpha = 0$$

Now we make some definitions.

A form  $\omega \in \Omega^r(M)$  is

<u>closed</u>	if	$d\omega = 0$
<u>coclosed</u>	if	$d^+ \omega = 0$
<u>exact</u>	if	<del><math>\omega = d\psi</math></del> $\omega = d\psi$ , some $\psi \in \Omega^{r-1}(M)$
<u>coexact</u>	if	$\omega = d^+ \psi$ , some $\psi \in \Omega^{r+1}(M)$
<u>harmonic</u>	if	$\Delta \omega = 0$

Note that by the above  $\omega$  is harmonic if and only if it is both closed and coclosed. Actually there is an interesting set of relationships among the spaces of the different kinds of forms. Define:

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$$C = \{ \text{closed } r\text{-forms} \} = Z^r(M)$$

$$CC = \{ \text{coclosed } r\text{-forms} \}$$

$$E = \{ \text{exact } r\text{-forms} \} = B^r(M) \subseteq Z^r(M)$$

$$CE = \{ \text{coexact } r\text{-forms} \}$$

$$H = \{ \text{harmonic } r\text{-forms} \} = \text{Harm}^r(M).$$

Then it turns out that  $\Omega^r(M)$  can be decomposed into 3 orthogonal subspaces:

$$\boxed{\Omega^r(M) = E \oplus CE \oplus H}$$

Proof: First show that spaces are orthogonal.

prove

$$(a) \langle E, CE \rangle = 0. \quad \text{let } \alpha = d\psi, \beta = d^+ \phi \quad (\alpha, \beta \in \Omega^r(M)).$$

$$\text{Then } \langle \alpha, \beta \rangle = \langle d\psi, d^+ \phi \rangle = \langle \psi, d^+ d^+ \phi \rangle = 0.$$

prove

$$(b) \langle E, H \rangle = 0. \quad \text{let } \alpha = d\psi, \Delta \beta = 0. \quad \text{Then}$$

$$\langle \alpha, \beta \rangle = \langle d\psi, \beta \rangle = \langle \psi, d^+ \beta \rangle = 0 \quad \text{since } \Delta \beta = 0 \Rightarrow d^+ \beta = 0.$$

$$(c) \text{ prove } \langle CE, H \rangle = 0. \quad \text{let } \alpha = d^+ \psi, \Delta \beta = 0. \quad \text{Then}$$

$$\langle \alpha, \beta \rangle = \langle d^+ \psi, \beta \rangle = \langle \psi, d \beta \rangle = 0 \quad \text{since } \Delta \beta = 0 \Rightarrow d \beta = 0.$$

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Next show that  $E \oplus CE \oplus H$  is the entire space  $\Omega^r(M)$ , by showing that if  $\omega \in \Omega^r(M)$  is orthogonal to  $E, CE$ , and  $H$ , then  $\omega = 0$ . This is a completeness proof. ~~Let's do it~~ ~~11/18/08~~ ~~CE~~ ~~H~~ suppose

$$(a) \quad \langle \omega, \alpha \rangle = 0 \quad \forall \alpha \in E, \text{ i.e., } \forall \alpha \text{ such that } \alpha = d\psi$$

$$(b) \text{ and } \langle \omega, \beta \rangle = 0 \quad \forall \beta \in CE, \text{ i.e., } \forall \beta \text{ such that } \beta = d^+ \phi$$

$$(c) \text{ and } \langle \omega, \gamma \rangle = 0 \quad \forall \gamma \in H, \text{ i.e., } \forall \gamma \text{ such that } \Delta \gamma = 0.$$

$$(a) \Rightarrow \langle \omega, d\psi \rangle = 0 \quad \forall \psi \in \Omega^{r-1}(M)$$

$$\Rightarrow \langle d^+ \omega, \psi \rangle = 0 \Rightarrow d^+ \omega = 0.$$

$$(b) \Rightarrow \langle \omega, d^+ \phi \rangle = 0 \quad \forall \phi \in \Omega^{r+1}(M)$$

$$\Rightarrow \langle d\omega, \phi \rangle = 0 \Rightarrow d\omega = 0$$

$$(c) \quad (a) \text{ and } (b) \Rightarrow d\omega = 0, \text{ so } (c) \Rightarrow \langle \omega, \omega \rangle = 0 \quad (\text{by } \gamma = \omega)$$

$$\Rightarrow \omega = 0.$$

QED.

see drawing next  
page.

Here are more relations. We know that  $E \subseteq C$  ( $B^r(M) \subseteq Z^r(M)$ ).

It turns out that  $C = E \oplus H$ . Similarly,  $CC = CE \oplus H$

Proof that  $C = E \oplus H$ . Let  $\langle \alpha, CE \rangle = 0$ , i.e.,  $\langle \alpha, d^+ \beta \rangle = 0, \forall \beta \in \Omega^{r+1}(M)$ . This implies  $\langle d\alpha, \beta \rangle = 0, \forall \beta, \Rightarrow d\alpha = 0$ . Conversely,  $d\alpha = 0 \Rightarrow \langle d\alpha, \beta \rangle = 0, \forall \beta \Rightarrow \langle \alpha, d^+ \beta \rangle = 0, \forall \beta \Rightarrow \langle \alpha, CE \rangle = 0$ . So  $\alpha$  is orthogonal to all co-exact forms iff  $\alpha$  is closed. This means,

$$C = E \oplus H$$

Sim.

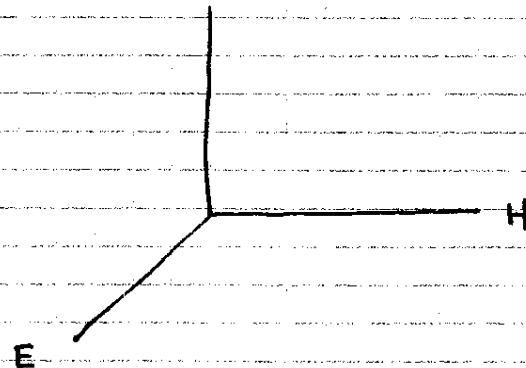
$$CC = CE \oplus H$$

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The Hilbert space  $\Omega^r(M)$ :

CE

 $C = "E-H \text{ plane}"$  $CC = "CE-H \text{ plane}"$ Hence  $H = C \cap CC$  as noted above.

From this follows a theorem. An arbitrary form  $\omega \in \Omega^r(M)$  has a unique decomposition,

$$\omega = \alpha + \beta + \gamma$$

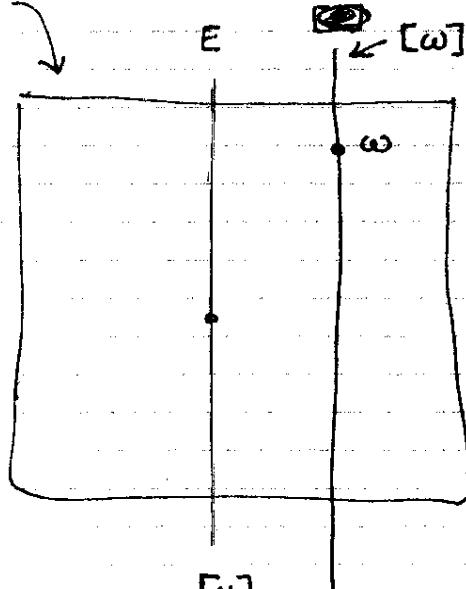
where  $\alpha = d\psi$ ,  $\beta = d^+ \phi$ ,  $\Delta \gamma = 0$ , i.e.,

$$\omega = d\psi + d^+ \phi + \gamma.$$

Finally, there are some connections with cohomology theory.

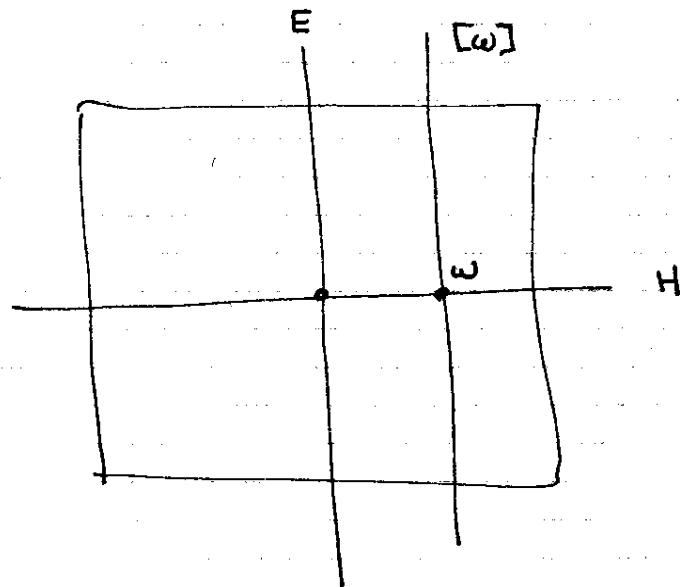
Recall, an element of  $H^r(M)$  is an equivalence class  $[\omega] = [\omega + df]$  of closed forms,  $d\omega = 0$ . Look at the geometry of the space  $Z^r(M) \stackrel{C}{=} \text{before}$  we put in a metric. It's just a vector space, sketch here as 2D, with a subspace of exact forms  $E = B^r(M)$ , sketch here as a 1D subspace:

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 $C = \text{whole plane}$ 

Then an equivalence class is seen geometrically as a plane (line) parallel to  $E$ , containing representative element  $\omega$ , see picture.

Now when we add a metric, we can talk about the orthogonal space in  $C$ , which is  $H = H^*(M)$ :



and there appears a privileged choice for a representative element of  $C$ :  
~~is element of  $C$~~ , a cohomology class, namely, a harmonic form:

Every cohomology class  $\in H^*(M)$  contains a unique harmonic form  $w$ . This is the form in the cohomology class that minimizes  $\langle w, w \rangle$ .

(a)

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An immediate corollary is that  $\text{Harm}^r(M)$  is isomorphic to  $H^r(M)$ ,

$$\boxed{\text{Harm}^r(M) \cong H^r(M)}$$

Every harmonic form corresponds to a unique cohomology class, and vice-versa.

The space of harmonic forms is otherwise the space of eigen-forms of  $\Delta$  with eigenvalue 0. Thus, the Betti number satisfies

$$b_r = \dim H^r(M) = \text{degeneracy of eigenvalue 0 of } \Delta \text{ acting on } \Omega^r(M).$$

Examples. For a zero form,  $\Delta f = 0 \Rightarrow df = 0 \Rightarrow f = \text{const}$ , and conversely. Thus (assuming  $M$  is connected),  $\text{Harm}^0(M)$  is spanned by  $f=1$ , it is one-dimensional, and  $b_0(M) = 1$ , which we knew already.

For case  $r=1$ , take some sample spaces.

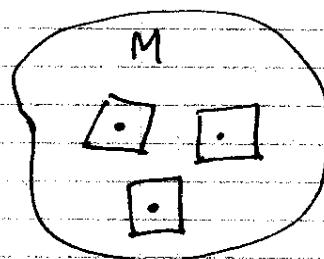
$$S^1 = \text{Circle}, \quad \Delta \omega = 0 \Rightarrow \omega = d\theta. \quad b_1 = 1$$

$$T^2 = \text{Torus}, \quad \Delta \omega = 0 \Rightarrow \omega = d\theta_1, d\theta_2. \quad b_1 = 2$$

$$S^2 = \text{sphere} \quad \Delta \omega = 0 \Rightarrow \omega = 0. \quad b_1 = 0.$$

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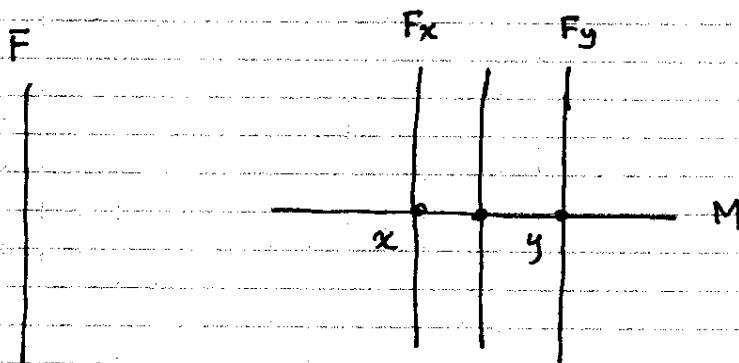
We turn now to fiber bundles. Begin with the intuitive idea, which is that a fiber bundle is a space made up by attaching identical copies of one space (the fiber) to each point of another space (the base space). For example, the tangent bundle  $TM$  to a manifold  $M$  is made up by attaching identical copies of  $\mathbb{R}^m$  ( $m = \dim M$ ) to each point  $x \in M$ . The copies are the tangent spaces  $T_x M$ .



Here  $M = \text{base space}$

std fiber =  $\mathbb{R}^m = F$ .

Sometimes we sketch a fiber bundle as if  $F$  (the std fiber) and  $M$  (the base space) were one-dimensional, since it's hard to draw higher dimensions.



( $M$  is the base space.)

Here  $F$  is the standard fiber, while  $F_x$  is the fiber over  $x$ ,  $x \in M$ .  $F_x$  is required to be diffeomorphic to  $F$  (this is what we meant by "identical" copy.)

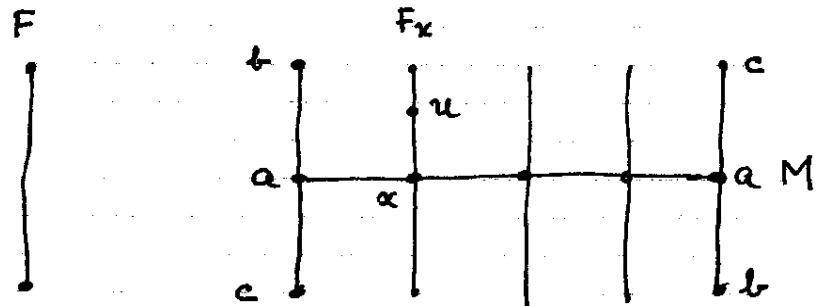
A fiber bundle is a way of creating a new space out of old spaces (here  $M$  and  $F$ ). It is a generalization of the cartesian product. In fact,  $M \times F$  is a fiber bundle (it is a way of attaching copies of  $F$  to points of  $M$ ), but most fiber bundles

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are not, (i.e., not diffeomorphic to)  $M \times F$ . All fiber bundles, are, however, locally diffeo. to such a Cartesian product. That is, if  $U \subset M$  is a sufficiently small open subset of  $M$ , then the set of fibers over  $U$  is diffeo to  $U \times F$ . (This is sort of like the idea that a sufficiently small region of a differentiable manifold is diffeo. to a region of  $\mathbb{R}^n$ , the coordinate space.) The reason most fiber bundles are not globally diffeo. to  $M \times F$  is that the different patches are fitted together with a kind of "twist".

The example of the Möbius strip will make this clear. The Möbius strip is the only <sup>nontrivial</sup> fiber bundle that is easy to visualize as a subset of  $\mathbb{R}^3$ . Here  $M = S^1$  (the circle) and  $F = [-1, 1]$ , a closed interval. Denote  $M$  by a line segment  $aa'$  with ends identified.



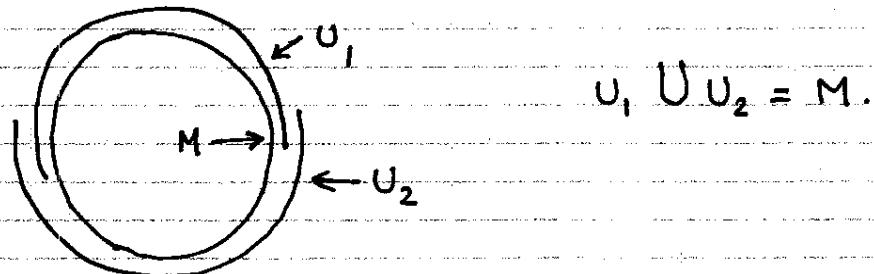
Because of the twist in the Möbius strip, the two fibers  $bc$ ,  $c'b$  over  $a$  are identified upside down. ~~All good~~

The entire space, the set of all fibers over all  $x \in M$ , is denoted  $E$  (the entire space), or the bundle for short. If  $u \in E$  is some point, it must belong to some fiber  $F_x$  over a point  $x \in M$ . Thus we define the projection map,

$$\pi: E \rightarrow M: u \mapsto x.$$

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The set of fibers over  $U \subset M$  is  $\pi^{-1}(U)$ . If  $U$  is small enough,  $\pi^{-1}(U) \cong U \times F$  ( $\cong$  means, "is diffeomorphic to"). In the case of the Möbius strip,  $U$  need only be any proper subset of  $M$ . Let us cover  $M = S^1$  with two open intervals, as illustrated:

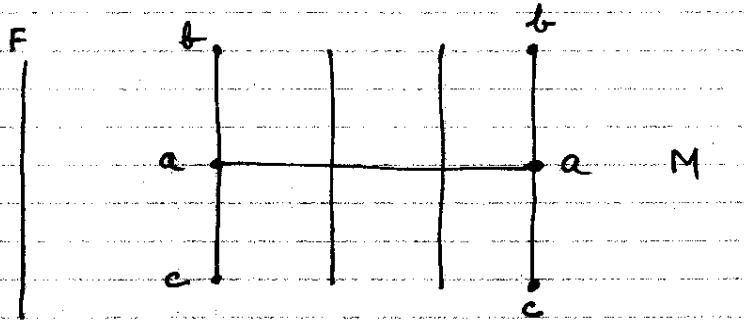


$$U_1, U_2, U_1 \cup U_2 = M.$$

Over each of  $U_1, U_2$ , the Möbius strip looks like a Cartesian product, but a twist is introduced in the overlaps that gives  $E$  a twisted product structure.

In general, a fiber bundle is said to be trivial iff  $E \cong M \times F$ . Triviality is a topological designation.

In the case of the Möbius strip, we could leave out the twist and we would get a cylinder:



In this case,  $Cyl = S^1 \times F$  (the bundle is trivial). The Möbius strip and cylinder illustrate the fact that two bundles can have the same base space and same fiber but not be identical. In fact, a problem in fiber bundle theory is to classify all bundles with a given

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M and F.

Now consider the tangent bundle. Some of this material was covered before. The tangent bundle  $TM$  has  $M$  = base space,  $F = \mathbb{E}$  standard fiber  $= \mathbb{R}^m$ ,  $E = \text{entire space} = TM$ . The tangent bundle is defined by

$$TM = \bigcup_{x \in M} T_x M.$$

It is the collection of all tangent vectors attached to all possible points  $x \in M$ . The projection map  $\pi: TM \rightarrow M$  is defined by  $\pi(T_x M) = x$ .

Before investigating the fiber bundle aspects of  $TM$  let's first prove that it is actually a differentiable manifold. (It is defined above as a collection of objects.) To do this, we must exhibit coordinates on  $TM$ . We are assuming  $M$  is a diff. manifold, so it possesses an open cover  $\{U_i\}$  with associated charts and coordinates. Let  $x^k$  be the coordinates on  $U \in \{U_i\}$ . Then  $\{e_\mu = \partial/\partial x^\mu\}$  is a frame at each  $x \in U$ , so any  $v \in T_x M$  can be written

$$v = v^\mu e_\mu|_x,$$

and we can take  $(x^\mu, v^\mu)$  as coordinates on  $\pi^{-1}(U)$ . Notice that  $\{\pi^{-1}(U_i)\}$  makes an open cover of  $TM$ , so coordinates like the above make an atlas on  $TM$ .  $TM$  is a differentiable manifold of dimension  $2m$ . In  $(x^\mu, v^\mu)$ , the  $x^\mu$  tell you which fiber you are on, and the  $v^\mu$  tell you where you are on the fiber.

The notion of a vector field can be given an interpretation in terms of the bundle  $TM$ . A vector field is of course a smooth assignment of a vector to each point  $x \in M$ . We can define it as a map,

$$X: M \rightarrow TM$$

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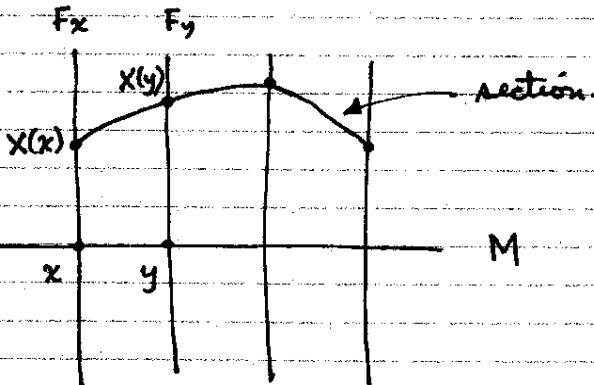
with the property,

$$\pi(X(x)) = x, \quad \forall x \in M,$$

which guarantees that  $X(x)$  is actually attached to  $x$ .

We can visualize a vector field geometrically as a submanifold of  $TM$ .

If we just plot the points  $X(x)$  in  $TM$  for  $x \in M$ , we get a surface:



The surface is called a section of the fiber bundle. Exactly the same picture works for any fiber bundle, a section is a ~~map  $S: M \rightarrow E$~~ <sup>global</sup> map  $S: M \rightarrow E$  such that  $\pi(S(x)) = x$ , or rather, the section is the image of this map. The section is a surface submanifold of  $TM$ , diffeomorphic to  $M$ , which intersects each fiber in one point.

The above is a global section. One can also discuss local sections, which are only defined over some subsets of  $M$ . For some bundles, global sections do not exist.

Similar to the tangent bundle  $TM$  is the cotangent bundle  $T^*M$ . It is the union of all the cotangent spaces  $T_x^*M$ . Similarly we can define the bundle of all type  $(0,2)$  tensor fields, etc. etc. There is a bundle for each type of tensor field. These are all examples of vector bundles, i.e., bundles in which the standard fiber is a vector space.

Another type of vector bundle occurs when  $M$  is a submanifold of Euclidean  $\mathbb{R}^n$  ( $m = \dim M < n$ ). Consider  $x \in M$  and the

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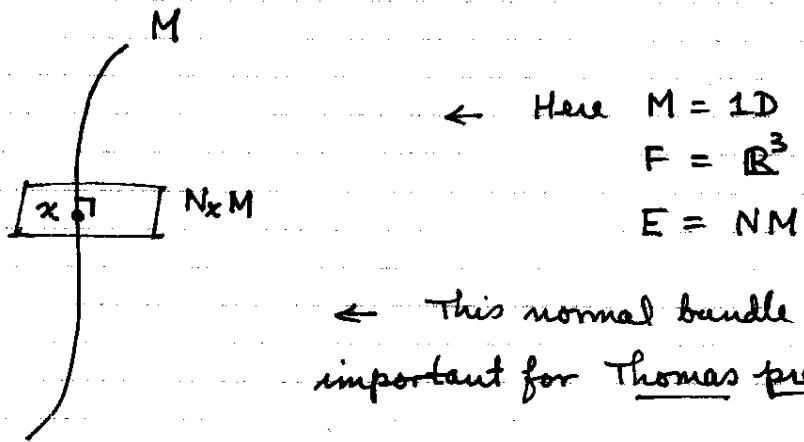
tangent space  $T_x M$ . This is a subspace of  $T_x (\mathbb{R}^n)$ .

Define  $N_x M$  as the normal space, i.e., the subspace of  $T_x (\mathbb{R}^n)$  complementary to and orthogonal to  $T_x M$ , so that

$$T_x (\mathbb{R}^n) = T_x M \oplus N_x M$$

and  $T_x M \perp N_x M$ .

For example, if  $M$  is the world line of a particle in Minkowski space-time,  $\mathbb{R}^4$ , then the normal space  $N_x M$  is the 3D space



of purely ~~spacetime~~<sup>spatial</sup> vectors in the rest frame of the particle at  $x$ . The ~~normal~~ normal bundle is then defined by

$$NM = \bigcup_{x \in M} N_x M.$$

Also Another bundle of interest is the frame bundle to a manifold  $M$ . Let  $x \in M$ , and let  $\{e_\mu\}$  be a frame at  $x$  (a set of ~~on~~<sup>linearly</sup> basis in  $T_x M$ ). We wish to construct a fiber  $F_x$  that will consist of all frames at  $x$ . If  $\{f_\nu\}$  is another frame, it is related to  $\{e_\mu\}$  by a nonsingular matrix, (a real,  $m \times m$  matrix)

$$f_\nu = \sum e_\nu A^\nu_\mu, \quad \text{where } \det(A) \neq 0.$$

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That is,  $A \in GL(m, \mathbb{R})$ . Thus (once the reference frame  $\{e_\mu\}$  is chosen) frames in  $F_x$  are placed in one-to-one correspondence with elements of  $G = GL(m, \mathbb{R})$ , and the standard fiber is  $F = G = GL(m, \mathbb{R})$ . Then the frame bundle  $FM$  is defined by

$$FM = \bigcup_{x \in M} F_x,$$

and  $\pi: FM \rightarrow M$  is defined in the usual way.

Frame bundles do not in general possess global sections. We say that  $M$  is parallelizable if  $FM$  possesses a global section. This would mean that we could construct  $m$  vector fields  $\{e_\mu\}$  on  $M$  that were everywhere linearly independent. In particular, none of the  $e_\mu$  could vanish anywhere.

A theorem states that the frame bundle  $FM$  is trivial iff it possesses a global section. Another theorem states that if  $FM$  is trivial, then so is  $TM$  (and  $T^*M$  and all the other tensor bundles). These are easy to prove, but we'll wait until we have proper definitions.

The frame bundle is an example of a principal fiber bundle.

A principal fiber bundle is one in which  $F = G$  = the structure group (defined later).

A vector bundle always possesses a global section, it is just the zero-section, for which the vector  $0$  is assigned to each  $x \in M$ .

Every vector space has a zero, so this is meaningful.

A more challenging question is to ask if a vector bundle possesses a section that vanishes nowhere. Let's talk about  $TM$  and vector

field. A zero of a vector field, a point  $x$  where  $X(x)=0$ , is usually regarded as a singularity of the vector field.  $X$  is smooth there (we are assuming we have only smooth fields), but the direction of  $X$  is not defined. Some manifolds do not possess a global, nonsingular (= nonzero) section of  $TM$ .

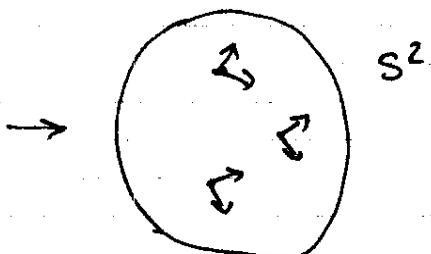
Here is a theorem (Poincaré-Hopf). It is discussed in Frankel but not Nakahara.

Theorem:  $TM$  possesses a global, nonvanishing (nowhere vanishing) section iff  $\chi(M)=0$ , where  $\chi$  is the Euler characteristic.

For example,  $\chi(S^2)=2$ , so there does not exist on  $S^2$  any smooth vector field that vanishes nowhere. This is the "hair on the coconut" theorem. But on the torus,  $\chi(T^2)=0$ , and such vector fields exist.

Since the vector fields that make up a ~~false~~ global section of the frame bundle vanish nowhere, we see that if  $\chi(M)\neq 0$  then the frame bundle is nontrivial, and has no global sections. For example, it is impossible to construct a smooth field of frames on  $S^2$ .

must have a discontinuity somewhere.



Such fields of frames are used in optics for polarization vectors of light waves.  $\vec{r}$  is the radial vector (in  $\vec{k}$ -space), and  $\hat{e}_1, \hat{e}_2$  are unit vectors (the polarization vectors) that span the plane  $\perp$  to  $\vec{r}$ . The books seldom mention that  $\hat{e}_1, \hat{e}_2$  cannot