11/13/08

Representations of the Lorentz group, in preparation for incorporating Dirac spinors into GR. First, the group $O(3,1)$ consists of matrices $\Lambda^{\mu}{}_{\cdot\nu}$ such that

$$\Lambda^{\mu}{}_{\cdot\nu}\,\Lambda^{\cdot\sigma}_{\mu} = \delta^{\sigma}_{\nu}$$

(raising and lowering with $\eta_{\mu\nu}$). Call the identity component of $O(3,1)$, L_0. It consists of Lorentz transformations that can be continuously connected with the identity. Physically, it L_0 contains Lorentz transformations that do <u>not</u> involve either parity or time reversal. Thus, $L_0 \subset O(3,1)$. Next, proper spatial rotations belong to $SO(3)$ which is a subgroup of L_0. And $SU(2)$ is the spin double cover of $SO(3)$. The spin double cover of

$$\begin{array}{ccccc}
SU(2) & \subset & SL(2,\mathbb{C}) & & \\
\downarrow \pi & & \downarrow \pi & & \\
SO(3) & \subset & L_0 & \subset & O(3,1)
\end{array}$$

L_0 is $SL(2,\mathbb{C})$, the set of all 2×2 complex matrices with unit determinant. $SU(2)$ is a subgroup of $SL(2,\mathbb{C})$.

There are 2 representations of $SL(2,\mathbb{C})$ we need. Let $g \in SL(2,\mathbb{C})$. One rep'n is $g \mapsto \Lambda(g)$, where $\Lambda \in L_0$ and $\Lambda(g)$ means the same as $\pi(g)$. The map π is not invertible, rather π^{-1} is double valued, just as with $SU(2)$ and $SO(3)$. Thus $g \mapsto \Lambda(g)$ is not a faithful representation. The 2nd representation is $g \mapsto D(g)$, where $D(g)$ are the 4×4 matrices that implement Lorentz transformations on Dirac spinors. This is a faithful representation of $SL(2,\mathbb{C})$, in fact in the right basis,

11/13/08

$$D(g) = \left(\begin{array}{c|c} g & 0 \\ \hline 0 & \bar{g} \end{array}\right) = SL(2,\mathbb{C}) \oplus \overline{SL(2,\mathbb{C})}.$$

The rep'n $D(g)$ has the following important properties, relative to the γ matrices of Dirac theory. These satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

which defines them up to a change of basis. Then we have

$$D(g)^{-1}\gamma^\mu D(g) = \Lambda(g)^\mu{}_\nu \gamma^\nu ,$$

which qualifies γ^μ as a 4-vector and intertwines the 2 representations $g \mapsto \Lambda(g)$ and $g \mapsto D(g)$. Next, we have

$$\gamma^0 D(g)^\dagger \gamma^0 = D(g)^{-1}.$$ (Matrices $D(g)$ are not unitary in general.)

If ψ is a Dirac spinor, then $D(g)\psi$ is interpreted as the Lorentz transformed (rotated, boosted, etc.) spinor.

Although $\pi: SL(2,\mathbb{C}) \to L_0$ is not an isomorphism, there is an isomorphism between the Lie algebras. This is the normal situation when one group is a cover of another: the Lie algebras are the same (isomorphic). Specifically, if $\Lambda = I + \Omega$ is a near-identity L.T. (i.e. Ω = small), then $\Omega_{\mu\nu} = -\Omega_{\nu\mu}$ (completely covariant version of Ω). Writing this as $\Lambda(g)$ where g is near-identity, the corresponding D matrix is

$$D(g) = 1 - \frac{i}{4}\Omega_{\mu\nu}\sigma^{\mu\nu},$$

where $\Omega_{\mu\nu}$ is a matrix of numbers and $\sigma^{\mu\nu}$ are Dirac matrices,

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu].$$

11/13/08

The matrices $\sigma^{\mu\nu}$ form a basis of the Lie algebra of the rep. $D(g)$.

To incorporate Dirac spinors into GR, we start with the Dirac Lagrangian in special relativity,

$$\mathcal{L}_D = \bar{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi$$

where $\partial_\mu = \partial/\partial x^\mu$ refers to the usual orthonormal Minkowski coordinates in S.R. There are 2 problems in incorporating this into G.R. The first is that the usual Dirac matrices γ^μ are linked to orthonormal coordinates in S.R., so how do we adapt them to curvilinear coordinates in GR? We sidestep this question by using an orthonormal vierbein in GR, call the forms θ^μ and dual basis vectors e_μ. Then we can use the standard γ^μ matrices of S.R. However, ∂_μ should be replaced by e_μ. We write $\partial_\mu\psi \to e_\mu\psi \equiv \psi_{,\mu}$.

Next, we must promote this ordinary derivative to a covariant derivative, i.e. replace $\psi_{,\mu} \to \nabla_\mu\psi$. How to define $\nabla_\mu\psi$?

We go back to the intuitive meaning of parallel transport between nearby points x and $x+\Delta x$, starting with a vector $Y \in T_xM$ that is parallel transported to $Y' \in T_{x+\Delta x}M$. Then we have

$$Y'^\mu = (\delta^\mu_\nu - \Delta x^\kappa\,\Gamma^\mu{}_{\kappa\nu})\,Y^\nu,$$

which in effect define the connection coefficients $\Gamma^\mu{}_{\kappa\nu}$. If we assume the basis θ^μ is orthonormal and the connection preserves the metric, $\nabla g = 0$, then the matrix ( ) above is

an infinitesimal Lorentz transformation, so 11/13/08

$$Y' = (I + \Omega) Y$$

where

$$\Omega_{\mu\nu} = -\Delta x^{\kappa} \Gamma_{\mu\kappa\nu},$$

where we have lowered the 1st index on Γ (with $\eta_{\mu\nu}$). Notice the role of the indices:

$$\Gamma_{\mu\kappa\nu}$$

1-form.

Lie algebra

$\Gamma_{\mu\kappa\nu} = -\Gamma_{\nu\kappa\mu}$ which means $\Omega_{\mu\nu} = -\Omega_{\nu\mu}$.

Now to parallel transport a Dirac spinor from x to $x+\Delta x$, we use the near-identity element of $D(g)$ corresponding to $I+\Omega$ above, that is,

$$\underset{\substack{\uparrow \\ @\ x+\Delta x}}{\psi'} = \left(1 - \frac{i}{4}\Omega_{\mu\nu}\sigma^{\mu\nu}\right)\underset{\substack{\uparrow \\ \text{at } x}}{\psi}$$

$$= \left(1 + \frac{i}{4}\Delta x^{\kappa}\Gamma_{\mu\kappa\nu}\sigma^{\mu\nu}\right)\psi.$$

This now leads to the covariant derivative,

$$\boxed{\nabla_{\kappa}\psi = \psi_{,\kappa} - \frac{i}{4}\Gamma_{\mu\kappa\nu}\sigma^{\mu\nu}\psi}$$

We incorporate this into the Lagrangian,

$$\mathcal{L}_D = \bar{\psi}\left(i\gamma^{\mu}\nabla_{\mu} - m\right)\psi$$

which can be added to the Einstein-Hilbert ~~action~~ Lagrangian in GR.

The terms in $\mathcal{L}_D$ involving Γ couple the Dirac particle to the gravitational field.

We should check that this Lagrangian is invariant under gauge transformations, i.e. local changes of orthonormal frames specified by $\{\theta^\mu\}$ or $\{e_\mu\}$. Let M = space-time, $U \subset M$ a region over which we carry out the gauge transformation, and let there be a map: $U \to SL(2,\mathbb{C}) : x \mapsto g(x)$. Then define

$$\Lambda(x) = \Lambda(g(x))$$
$$D(x) = D(g(x)).$$

Under the gauge transformation, the frames transform according to

$$\theta'^\mu\big|_x = \Lambda(x)^\mu{}_\nu \; \theta^\nu\big|_x$$

and tensors transform in the obvious way. $\Gamma_{\mu\kappa\nu}$ transforms in a more complicated manner, which was discussed in a recent lecture. The Dirac spinor transforms according to

$$\psi'(x) = D(x)\,\psi(x).$$

When we check that $\mathcal{L}_D$ is invariant under gauge transformations, the problem reduces to showing that $\nabla_\mu\psi$ transforms as a spinor in the Dirac spin index, and as a covector in the index μ, i.e.,

$$\nabla'_\mu \psi' = D(x)\,\Lambda(x)_\mu{}^\nu \; \nabla_\nu \psi .$$

This will be left as an exercise.

11/13/08

Now we begin Hodge * theory and harmonic forms. To preview the results a bit, when we add a metric to a manifold we can do new things with differential forms and find new connections to old subjects such as cohomology groups (which do not require a metric for their definition).

If we add a metric to M, we can define a scalar product of wave functions,

$$\cancel{\langle f,g\rangle = \int_M \sqrt{|g|}\, d^m x\, f g \qquad m = \dim M}$$

$$\langle f_1, f_2\rangle = \int_M d^m x \sqrt{|g|}\; f_1 f_2$$

where $m = \dim M$, $f_1, f_2 \in \mathcal{F}(M)$. (Real valued functions here.) Thus the wave functions make a Hilbert space. We also have interesting operators that act on these wave fns, such as the generalized Laplacian ∇^2 (which requires a metric for its definition), and which lead to orthonormal sets of eigenfunctions.

All this (the scalar product, Laplacians) etc. can be generalized to arbitrary r-forms. It turns out for example that the degeneracy of the 0 eigenvalue of ∇^2 is the same as the Betti number of M.

The <u>permutation</u> or <u>Levi-Civita symbol</u> is familiar:

$$\varepsilon_{\mu_1 \cdots \mu_m} = \begin{cases} +1 & (\mu_1 \cdots \mu_m) = \text{even pmut of } (1 \ldots m) \\ -1 & (\mu_1 \cdots \mu_m) = \text{odd pmut of } (1 \ldots m) \\ 0 & \text{otherwise} \end{cases}$$

Just because we put lower indices on it does not mean that it is a tensor. In fact, suppose a tensor has components $\varepsilon_{\mu_1 \ldots \mu_m}$ in one coord. system x^μ, and examine what its components are in another coord. syst. x'^μ:

11/13/08

$T_{\mu_1 \cdots \mu_m} = \varepsilon_{\mu_1 \cdots \mu_m}$ in coords x^μ (suppose).

Then in coords y^μ,

$$T'_{\mu_1 \cdots \mu_m} = \frac{\partial x^{\nu_1}}{\partial y^{\mu_1}} \cdots \frac{\partial x^{\nu_m}}{\partial y^{\mu_m}} \varepsilon_{\nu_1 \cdots \nu_m} = \det\left(\frac{\partial x}{\partial y}\right) \varepsilon_{\mu_1 \cdots \mu_m}.$$

So the ε-symbol does not transform as a tensor (and we shall not call it a tensor.) Now let $g_{\mu\nu}$, $g'_{\mu\nu}$ be the metric in coords x^μ and y^μ, so that

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu} g_{\alpha\beta},$$

or

$$g' = \left(\det \frac{\partial x}{\partial y}\right)^2 g \qquad \text{where} \qquad \begin{aligned} g &= \det g_{\mu\nu} \\ g' &= \det g'_{\mu\nu} \end{aligned}$$

so

$$\left| \det \frac{\partial x}{\partial y} \right| = \sqrt{\frac{|g'|}{|g|}}.$$

We put $|\ |$ around g, since it may be negative (depends on the signature, $\mathrm{sgn}(g) = -1$ in GR). Now we suppose M is orientable and we choose an orientation and work only with oriented atlases. Then $\det \frac{\partial x}{\partial y} > 0$, and we can drop the $|\ |$ around $\det \frac{\partial x}{\partial y}$. Then we see that if $T_{\mu_1 \cdots \mu_m} = \sqrt{|g|}\, \varepsilon_{\mu_1 \cdots \mu_m}$ in one coordinates, then $T'_{\mu_1 \cdots \mu_m} = \sqrt{|g'|}\, \varepsilon_{\mu_1 \cdots \mu_m}$ in another. We have a tensor, if we restrict to oriented coordinates. In fact it is an m-form, which we henceforth write as Ω:

$$\Omega = \frac{1}{m!} \sqrt{|g|}\, \varepsilon_{\mu_1 \cdots \mu_m}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_m}$$

i.e.

$$\Omega = \sqrt{|g|}\, dx^1 \wedge \cdots \wedge dx^m \qquad \text{(coord basis)}$$

or

$$\Omega = \sqrt{|g|}\, \theta^1 \wedge \cdots \wedge \theta^m \qquad \text{(any basis).}$$

11/13/08

Here $\{\theta^\mu\}$ is any basis (coordinate or non-coordinate). Note that if $\{\theta^\mu\}$ is an O.N. vielbein, then $\sqrt{|g|}=1$ and $\Omega = \theta^1\wedge\cdots\wedge\theta^m$.

It is of interest to compute the completely contravariant components of Ω:

$$\Omega^{\mu_1\cdots\mu_m} = g^{\mu_1\nu_1}\cdots g^{\mu_m\nu_m}\,\Omega_{\nu_1\cdots\nu_m}$$

$$= \det(g^{\mu\nu})\,\sqrt{|g|}\,\varepsilon_{\mu_1\cdots\mu_m}$$

But $\det g^{\mu\nu} = \frac{1}{g} = \mathrm{sgn}(g)/|g|$. So,

$$\boxed{\Omega^{\mu_1\cdots\mu_m} = \frac{\mathrm{sgn}(g)}{\sqrt{|g|}}\,\varepsilon_{\mu_1\cdots\mu_m}.}$$ useful later.

We don't worry that LHS has upper indices and RHS has lower, since ε is not a tensor.

Ω is called the <u>invariant volume form</u> since its integral over any region $R\subseteq M$ is the volume of that region in the metrical sense,

$$\int_R \Omega = \mathrm{vol}(R).$$

On a space with $m=\dim M$ dimensions, both r-forms and $(m-r)$-forms have the same number of indep. components,

$$\binom{m}{r} = \binom{m}{m-r}.$$

Thus r-forms and $(m-r)$-forms (at a point $x\in M$) are vector spaces of the same dimensionality, and are isomorphic as vector spaces.

11/13/08

In the absence of a metric or other additional structure, however, there is no <u>natural</u> isomorphism between these spaces. Now, however, we will assume we have a metric (M,g). Then there is a natural mapping, an isomorphism, actually, between these space,

$$\text{Hodge } * : \Omega^r(M) \to \Omega^{m-r}(M).$$

It is defined by its action on the basis forms of $\Omega^r(M)$, then extended to arb. r-forms by linearity. The defn. is

$$\boxed{* \left(\theta^{\mu_1} \wedge \dots \wedge \theta^{\mu_r}\right) = \frac{1}{(m-r)!} \Omega^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_{m-r}} \left(\theta^{\nu_1} \wedge \dots \wedge \theta^{\nu_{m-r}}\right).}$$

Indices on Ω are raised with $g^{\mu\nu}$.

As a special case, consider the 0-form $1 \in \Omega^0(M)$ (const scalar). Then $r=0$ in the above, and we have

$$* 1 = \frac{1}{m!} \Omega_{\nu_1 \dots \nu_m} \theta^{\nu_1} \wedge \dots \wedge \theta^{\nu_m} = \Omega,$$

$$\boxed{* 1 = \Omega}$$

The defn above makes it clear that $*$ is linear, but is it an isomorphism (i.e., is it invertible)? We answer by computing $**$, a map: $\Omega^r(M) \to \Omega^r(M)$. We apply defn above twice, get

$$** \left(\theta^{\mu_1} \wedge \dots \wedge \theta^{\mu_r}\right) = \frac{1}{(m-r)!} \Omega^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_{m-r}}$$

$$\times \frac{1}{r!} \Omega^{\nu_1 \dots \nu_{m-r}}{}_{\lambda_1 \dots \lambda_r} \left(\theta^{\lambda_1} \wedge \dots \wedge \theta^{\lambda_r}\right).$$

11/13/08

Transform this. First raise + lower $\nu_1 \dots \nu_{m-r}$ indices to make indices uniformly upper or lower. Next, on $\Omega_{\nu_1 \dots \nu_{m-r} \lambda_1 \dots \lambda_r}$, migrate λ indices to left of ν indices. This involves $(m-r)r$ sign changes, so

$$\Omega_{\nu_1 \dots \nu_{m-r} \lambda_1 \dots \lambda_r} = (-1)^{r(m-r)} \Omega_{\lambda_1 \dots \lambda_r \nu_1 \dots \nu_{m-r}}.$$

Thus,

$$** (\theta^{\mu_1} \wedge \dots \wedge \theta^{\mu_r}) = \frac{1}{(m-r)!} \frac{1}{r!} (-1)^{r(m-r)}$$

$$\times \boxed{\Omega^{\mu_1 \dots \mu_r \nu_1 \dots \nu_{m-r}} \Omega_{\lambda_1 \dots \lambda_r \nu_1 \dots \nu_{m-r}}} \theta^{\lambda_1} \wedge \dots \wedge \theta^{\lambda_r}$$

$$= \frac{\mathrm{sgn}(g)}{\sqrt{|g|}} \varepsilon_{\mu_1 \dots \mu_r \nu_1 \dots \nu_{m-r}} \times \sqrt{|g|} \, \varepsilon_{\lambda_1 \dots \lambda_r \nu_1 \dots \nu_{m-r}}$$

$$= \mathrm{sgn}(g) \, \mathrm{sgn}\begin{pmatrix} \mu_1 \dots \mu_r \\ \lambda_1 \dots \lambda_r \end{pmatrix} (m-r)!$$

where we use identities for products of two ε's and where

$$\mathrm{sgn}\begin{pmatrix} \mu_1 \dots \mu_r \\ \lambda_1 \dots \lambda_r \end{pmatrix} = \begin{cases} \pm 1 \text{ if } (\lambda_1 \dots \lambda_r) \text{ is } \binom{\text{even}}{\text{odd}} \text{ perm of } \mu_1 \dots \mu_r \\ 0 \quad \text{otherwise.} \end{cases}$$

Thus,

$$\mathrm{sgn}\begin{pmatrix} \mu_1 \dots \mu_r \\ \lambda_1 \dots \lambda_r \end{pmatrix} \theta^{\lambda_1} \wedge \dots \wedge \theta^{\lambda_r}$$

$$= r! \; \theta^{\mu_1} \wedge \dots \wedge \theta^{\mu_r}.$$

Putting it all together, we have

$$** \left(\theta^{\mu_1}\wedge\cdots\wedge\theta^{\mu_r}\right) = \mathrm{sgn}(g)\,(-1)^{r(m-r)}\left(\theta^{\mu_1}\wedge\cdots\wedge\theta^{\mu_r}\right).$$

or

$$\boxed{** = \mathrm{sgn}(g)\,(-1)^{r(m-r)}}$$ when acting on $\omega\in\Omega^r(M)$.

equivalently,

$$\boxed{*^{-1} = \mathrm{sgn}(g)\,(-1)^{r(m-r)}\,*}$$

Thus $*$ is invertible, and $*$ is an isomorphism.

Now consider the interaction of $*$ with $\wedge$. Let $\alpha,\beta\in\Omega^r(M)$. Then $*\beta$ is an $(m-r)$-form, and

$$\alpha\wedge *\beta \in \Omega^m(M).$$

Thus $\alpha\wedge*\beta$ must be proportional to the volume form Ω, i.e., it must be $f\Omega$ for some scalar f. Now we work out what f is.

Write

$$\alpha = \frac{1}{r!}\alpha_{\mu_1\cdots\mu_r}\,\theta^{\mu_1}\wedge\cdots\wedge\theta^{\mu_r}$$

$$\beta = \frac{1}{r!}\beta_{\nu_1\cdots\nu_r}\,\theta^{\nu_1}\wedge\cdots\wedge\theta^{\nu_r}$$

Hence

$$*\beta = \frac{1}{r!}\beta_{\nu_1\cdots\nu_r}\frac{1}{(m-r)!}\Omega^{\nu_1\cdots\nu_r}{}_{\lambda_1\cdots\lambda_{m-r}}\,\theta^{\lambda_1}\wedge\cdots\wedge\theta^{\lambda_{m-r}},$$

and

$$\alpha\wedge*\beta = \frac{1}{(r!)^2(m-r)!}\alpha_{\mu_1\cdots\mu_r}\beta_{\nu_1\cdots\nu_r}\Omega^{\nu_1\cdots\nu_r}{}_{\lambda_1\cdots\lambda_{m-r}}$$
$$\times\ \theta^{\mu_1}\wedge\cdots\wedge\theta^{\mu_r}\wedge\theta^{\lambda_1}\wedge\cdots\wedge\theta^{\lambda_{m-r}}.$$

11/13/08

Transform this, First raise + lower $\nu_1 \dots \nu_r$ indices, use $\theta^{\mu_1} \wedge \dots \wedge \theta^{\mu_r} \wedge \theta^{\lambda_1} \wedge \dots \wedge \theta^{\lambda_{m-r}} = \varepsilon_{\mu_1 \dots \mu_r \lambda_1 \dots \lambda_{m-r}} \; \theta^1 \wedge \dots \wedge \theta^m$. Get,

$$\alpha \wedge * \beta = \frac{1}{(r!)^2 (m-r)!} \alpha_{\mu_1 \dots \mu_r} \beta^{\nu_1 \dots \nu_r} \sqrt{|g|} \, \varepsilon_{\nu_1 \dots \nu_r \lambda_1 \dots \lambda_{m-r}}$$

$$\times \; \varepsilon_{\mu_1 \dots \mu_r \lambda_1 \dots \lambda_{m-r}} \; \theta^1 \wedge \dots \wedge \theta^m$$

$$= \frac{1}{(r!)^2 (m-r)!} \alpha_{\mu_1 \dots \mu_r} \beta^{\nu_1 \dots \nu_r} (m-r)! \, \mathrm{sgn}\begin{pmatrix} \mu_1 \dots \mu_r \\ \nu_1 \dots \nu_r \end{pmatrix} \Omega$$

$$\boxed{\alpha \wedge * \beta = \left(\frac{1}{r!} \alpha_{\mu_1 \dots \mu_r} \beta^{\mu_1 \dots \mu_r} \right) \Omega}$$

The scalar multiplying Ω is the complete contraction of the components of α with those of β.

Several things to note about this. First, the answer is symmetric in α, β, so

$$\boxed{\alpha \wedge * \beta = \beta \wedge * \alpha}.$$

Next, if g is pos. def., then $\alpha_{\mu_1 \dots \mu_r} \alpha^{\mu_1 \dots \mu_r} \geq 0$, i.e. you get a pos. def. scalar product of r-forms. Define

$$\boxed{\langle \alpha, \beta \rangle = \int_M \alpha \wedge * \beta. \quad = \langle \beta, \alpha \rangle}$$

Then if g is pos. def. (a Riemannian manifold) then this scalar product is also pos. def., i.e., $\langle \alpha, \alpha \rangle \geq 0$ and $\langle \alpha, \alpha \rangle = 0$ iff $\alpha = 0$.

Notice that if α, β are 0-forms (call them f_1, f_2), then we get 11/13/08 the obvious scalar product of them,

$$\langle f_1, f_2 \rangle = \int \Omega\, f_1 f_2 = \int d^m x \sqrt{|g|}\, f_1 f_2 .$$

More generally, if g is pos. def., we have a scalar product on $\Omega^r(M)$ that allows us to define a <u>Hilbert space of r-forms</u>. The functional analysis of this is easiest in the case of compact M.

An example of this scalar product. Let F be the EM field tensor in 4D space time, (maybe curved),

$$F = \frac{1}{2} F_{\mu\nu}\, \theta^\mu \wedge \theta^\nu .$$

Then

$$\langle F, F \rangle = \int F \wedge * F = \frac{1}{2} \int F_{\mu\nu} F^{\mu\nu} \sqrt{|g|}\, d^4 x .$$

This is 2x the EM field action,

$$S_{EM} = \frac{1}{2} \langle F, F \rangle .$$

(But the scalar product is not pos. def. on space-time.)

Now consider interaction of $*$ with exterior deriv. d.

Let $\alpha \in \Omega^r(M)$, $\beta \in \Omega^{r-1}(M)$. Then $\langle \alpha, d\beta \rangle$ is meaningful.

We define the operator $d^\dagger$ (the adjoint of d) by

$$\langle \alpha, d\beta \rangle = \langle d^\dagger \alpha, \beta \rangle, \qquad \forall\, \alpha \in \Omega^r(M),\ \beta \in \Omega^{r-1}(M).$$

$d^\dagger$ is the unique operator that makes this equation true. Note that

$$d^\dagger : \Omega^r(M) \to \Omega^{r-1}(M)$$
$$d : \Omega^{r-1}(M) \to \Omega^r(M)$$

11/13/08

(d and d^+ work in opposite directions).

We can find an expression for d^+ as follows.

$$\langle d^+\alpha, \beta\rangle = \langle \alpha, d\beta\rangle = \langle d\beta, \alpha\rangle = \int_M d\beta \wedge *\alpha.$$

But $d(\beta\wedge *\alpha) = d\beta \wedge *\alpha + (-1)^{r-1}\beta\wedge d*\alpha$, so

$$= \int_M d(\beta\wedge *\alpha) - (-1)^{r-1}\int_M \beta\wedge d*\alpha.$$

First term vanishes by Stokes' thm (we assume $\partial M = 0$), so

$$= (-1)^r \int_M \beta\wedge d*\alpha = (-1)^r \int_M \beta\wedge *(*^{-1}d*\alpha)$$

$$= (-1)^r \langle \beta, *^{-1}d*\alpha\rangle = (-1)^r \langle *^{-1}d*\alpha, \beta\rangle.$$

This implies,

$$\boxed{d^+ = (-1)^r *^{-1} d *}$$ acting on r-forms.

In this expression, $*^{-1}$ acts on an $(m-r+1)$-form, so

$$*^{-1} = \mathrm{sgn}(g)\,(-1)^{(m-r+1)(r-1)} *.$$

Since

$$r + (m-r+1)(r-1) \equiv mr + m + 1 \pmod 2,$$

we have

$$\boxed{d^+ = \mathrm{sgn}(g)\,(-1)^{mr+m+1} * d *}$$ alternative expression, acting on r-forms.

11/13/08

We note the identity,

$$\boxed{d^+ d^+ = 0}$$

which is easily proved,

sign of $**$.

$$d^+ d^+ = *d**d* = \pm *dd* = 0.$$

Note that d^+ annihilates any 0-form,

$$d^+ f = 0, \qquad f \in \mathcal{F}(M)$$

because there are no (-1)-forms.

11/13/08

Summary:

$$\Omega = \sqrt{|g|}\ \theta^1 \wedge \dots \wedge \theta^m = \frac{\sqrt{|g|}}{m!}\ \varepsilon_{\mu_1 \dots \mu_m}\ \theta^{\mu_1} \wedge \dots \wedge \theta^{\mu_m}$$

$$\Omega_{\mu_1 \dots \mu_m} = \sqrt{|g|}\ \varepsilon_{\mu_1 \dots \mu_m}$$

$$\Omega^{\mu_1 \dots \mu_m} = \frac{\mathrm{sgn}(g)}{\sqrt{|g|}}\ \varepsilon_{\mu_1 \dots \mu_m}$$

$$\Omega = *1$$

$$* : \Omega^r(M) \to \Omega^{m-r}(M)$$

$$*(\theta^{\mu_1} \wedge \dots \wedge \theta^{\mu_r}) = \frac{1}{(m-r)!}\ \Omega^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_{m-r}}\ (\theta^{\nu_1} \wedge \dots \wedge \theta^{\nu_{m-r}})$$

$$\left.\begin{aligned} ** &= \mathrm{sgn}(g)(-1)^{r(m-r)} \\ *^{-1} &= \mathrm{sgn}(g)(-1)^{r(m-r)}\, * \end{aligned}\right\} \text{ on } \omega \in \Omega^r(M)$$

$$\alpha \wedge *\beta = \left(\frac{1}{r!}\ \alpha_{\mu_1 \dots \mu_r}\ \beta^{\mu_1 \dots \mu_r}\right)\Omega, \qquad \alpha, \beta \in \Omega^r(M)$$

$$\alpha \wedge *\beta = \beta \wedge *\alpha$$

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta = \langle \beta, \alpha \rangle \qquad (\text{pos def. if } g \text{ pos def.})$$

$$\langle \alpha, d\beta \rangle = \langle d^+\alpha, \beta \rangle, \qquad \forall\ \alpha \in \Omega^r(M),\ \beta \in \Omega^{r-1}(M).$$

$$d^+ : \Omega^r(M) \to \Omega^{r-1}(M)$$

$$d : \Omega^r(M) \to \Omega^{r+1}(M)$$

$$d^+ = (-1)^r *^{-1} d * = \mathrm{sgn}(g)(-1)^{mr+m+1} * d *. \quad (\text{on } \omega \in \Omega^r(M))$$

$$d^+ d^+ = 0$$

11/13/08

Now work out the action of d^+ on a 1-form $\alpha \in \Omega^1(M)$. (We know d^+ annihilates 0-forms). $d^+\alpha$ is a scalar, want to find it. Write $\alpha = \alpha_\mu \theta^\mu$.

First compute $*\alpha$:

$$*\alpha = \frac{\alpha_\mu}{(m-1)!} \Omega^\mu{}_{\nu_2 \dots \nu_m} (\theta^{\nu_2} \wedge \dots \wedge \theta^{\nu_m}) \qquad \text{(Raise + lower } \mu\text{)}$$

$$= \frac{1}{(m-1)!} \alpha^\mu \underbrace{\boxed{\Omega_{\mu\nu_2 \dots \nu_m}}}_{\sqrt{|g|}\, \varepsilon_{\mu\nu_2 \dots \nu_m}} (\theta^{\nu_2} \wedge \dots \wedge \theta^{\nu_m})$$

$$d*\alpha = \frac{1}{(m-1)!} \left(\sqrt{|g|}\, \alpha^\mu\right)_{,\sigma} \varepsilon_{\mu\nu_2 \dots \nu_m} \underbrace{\theta^\sigma \wedge \theta^{\nu_2} \wedge \dots \wedge \theta^{\nu_m}}$$

$$= \varepsilon_{\sigma\nu_2 \dots \nu_m} \underbrace{\theta^1 \wedge \dots \wedge \theta^m}_{\frac{\Omega}{\sqrt{|g|}}}.$$

$$d*\alpha = \frac{1}{(m-1)!} \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|}\, \alpha^\mu\right)_{,\sigma} (m-1)!\, \mathrm{sgn}\begin{pmatrix}\sigma \\ \mu\end{pmatrix} \Omega$$

Note: $\mathrm{sgn}\begin{pmatrix}\sigma \\ \mu\end{pmatrix} = \delta^\sigma_\mu$. More generally,

$$\mathrm{sgn}\begin{pmatrix}\sigma_1 \dots \sigma_r \\ \mu_1 \dots \mu_r\end{pmatrix} = \begin{vmatrix} \delta^{\sigma_1}_{\mu_1} & \dots & \delta^{\sigma_1}_{\mu_r} \\ \vdots & & \vdots \\ \delta^{\sigma_r}_{\mu_1} & \dots & \delta^{\sigma_r}_{\mu_r} \end{vmatrix}$$

$$= \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|}\, \alpha^\mu\right)_{,\mu} \Omega.$$

A digression on a useful theorem. Let $X = X^\mu e_\mu$ be a vector field. Then 11/13/08

$$\boxed{X^\mu{}_{;\mu} = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}\, X^\mu\right)_{,\mu}.}$$ useful formula.

We used this theorem in computing Einstein field eqns. from Lagrangian.

To prove it, expand RHS,

$$\text{RHS} = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}\right)_{,\mu} X^\mu + X^\mu{}_{,\mu}$$

But by the formula for the derivative of a determinant,

$$= \frac{1}{2}\left(g^{\alpha\beta} g_{\alpha\beta,\mu}\right) X^\mu + X^\mu{}_{,\mu}.$$

Now
$$X^\mu{}_{;\nu} = X^\mu{}_{,\nu} + \Gamma^\mu_{\alpha\nu} X^\alpha,$$

so $\text{LHS} = X^\mu{}_{;\mu} = X^\mu{}_{,\mu} + \Gamma^\mu_{\alpha\mu} X^\alpha.$

Also,
$$\Gamma^\mu_{\alpha\beta} = \tfrac{1}{2} g^{\mu\nu}\left(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}\right),$$

so
$$\Gamma^\mu_{\alpha\mu} = \tfrac{1}{2} g^{\mu\nu}\left(\cancel{g_{\nu\alpha,\mu}} + g_{\nu\mu,\alpha} - \cancel{g_{\alpha\mu,\nu}}\right),$$

(two terms cancel by exchange $\mu \leftrightarrow \nu$ and symmetry)

$$\Gamma^\mu_{\alpha\mu} = \tfrac{1}{2} g^{\mu\nu} g_{\mu\nu,\alpha}.$$

so

$$\text{LHS} = X^\mu{}_{,\mu} + \tfrac{1}{2} g^{\mu\nu} g_{\mu\nu,\alpha} X^\alpha = \text{RHS}.$$ QED

So to go back, we have

$$d*\alpha = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}\,\alpha^{\mu}\right)_{,\mu}\Omega \quad = \quad \alpha^{\mu}{}_{;\mu}\,\Omega$$

Now apply $(-1)^r *^{-1} = -*^{-1}$, note that $\Omega = *1$ so $*^{-1}\Omega = 1$, get

$$d^{+}\alpha = -*^{-1} d*\alpha = -\alpha^{\mu}{}_{;\mu}.$$

$$\boxed{d^{+}\alpha = -\alpha^{\mu}{}_{;\mu}}$$

It is the covariant minus "divergence" of α (converted to a vector via g).

Note: in special case $\alpha = df$ $\quad (f \in \Omega^0(M))$,

we have

$$d^{+}df = -f^{;\mu}{}_{;\mu}.$$

This is (minus) the obvious generalization of the <u>Laplacian</u> to curved spaces,

$$-\nabla^2 f = -f_{,ii} \quad \text{on Euclidean } \mathbb{R}^n.$$

Another note on this result:

$$\int_M d^n x\sqrt{|g|}\;\alpha^{\mu}{}_{;\mu} = -\int_M (d^{+}\alpha)\Omega = -\int_M d^{+}\alpha \wedge *1$$

$$= -\langle d^{+}\alpha, 1\rangle = -\langle \alpha, d1\rangle = 0.$$

A more straightforward way to see the same thing is to use integration by parts,