\[
\frac{1}{2} \left( \Gamma^\mu_{\nu\sigma} - \Gamma^\mu_{\sigma\nu} \right) \theta^\nu \wedge \theta^\sigma \\
= \frac{1}{2} \left( R^\mu_{\nu\sigma} - \Gamma^\nu_{\beta\sigma} R^\mu_{\beta\nu} + \Gamma^\nu_{\alpha\sigma} R^\mu_{\beta\nu} \right) \theta^\nu \wedge \theta^\sigma \\
= R^\mu_{\nu} - \frac{1}{2} \omega^\nu_\sigma \wedge \omega^\sigma_\nu + \frac{1}{2} \omega^\nu_\sigma \wedge \omega^\nu_\sigma,
\]

\[\nabla \text{equal}\]

\[
d(\omega^\mu_\nu) + \omega^\mu_\sigma \wedge \omega^\sigma_\nu = R^\mu_{\nu}
\]

2nd Cartan structure eqn.

Again, take
\[
T^\mu = d\phi^\mu + \omega^\mu_\alpha \wedge \theta^\alpha,
\]
apply d,
\[
dT^\mu = 0 + (R^\mu_\alpha - \omega^\mu_\sigma \wedge \omega^\sigma_\alpha) \wedge \theta^\alpha
\]
\[
- \omega^\mu_\alpha \wedge (d\theta^\alpha) \quad T^\alpha - \omega^\alpha_\beta \wedge \theta^\beta
\]

1st Bianchi identity, generalized to case \( T \neq 0 \).

Finally, take \( \mathbb{R}^\infty \), 2nd Cartan structure, apply d:
\[
dR^\mu_{\nu} = d\omega^\mu_\sigma \wedge \omega^\sigma_\nu - \omega^\mu_\sigma \wedge d\omega^\sigma_\nu
\]
\[
= (R^\mu_\sigma - \omega^\mu_\alpha \wedge \omega^\alpha_\sigma) \wedge \omega^\sigma_\nu
\]
\[
- \omega^\mu_\sigma \wedge (R^\sigma_\nu - \omega^\sigma_\alpha \wedge \omega^\alpha_\nu)
\]
One might say that the covariant exterior derivative of the curvature 2-form is 0, so that the form is closed in this sense.

$$dR_{\mu \nu} + \Omega^\sigma \wedge R_{\sigma \nu} - R^\sigma \sigma \wedge \Omega_{\nu} = 0$$

2nd Bianchi, generalized.

When $T=0$, these again should reduce to the previous versions of the Bianchi identities. For the 1st Bianchi identity, this gives

$$0 = R^\mu_{\alpha \nu} \wedge \Theta^\alpha = \frac{1}{2} R^\mu_{\nu \rho \sigma} \Theta^\nu \wedge \Theta^\rho \wedge \Theta^\sigma$$

$$\Rightarrow R^\mu_{\nu \rho \sigma}[\nu \rho \sigma] = 0. \quad \text{checks.}$$

For the 2nd Bianchi identity, notice that it doesn't involve $T$ at all. But if you want to show equivalence to $R^\mu_{\nu}[\nu \rho \sigma] = 0$, you must use $T=0$.

Now consider the case that we have a metric $g$ and a metric connection $\nabla g = 0$.

Then it is convenient to assume the basis $\{e_\mu\}$ is orthonormal, i.e.,

$$g_{\mu \beta} = g(e_\mu, e_\beta) = \eta_{\mu \beta} = \eta_{\mu \beta} \quad \text{(pseudo-Riem. case, or } g_{\mu \beta}, \text{ Riem. case).}$$

$$= \text{const. metric of special relativity.}$$

We know that if the curvature tensor $\neq 0$, then there is no coordinate basis such that $g_{\mu \beta} = \eta_{\mu \beta}$. But there are always non-coordinated bases that make this true. This is a special kind of vielbein.
There are some special properties of $\Gamma, R$ in orthonormal vielbeins. First, $\nabla g = 0$ implies

$$0 = g_{\mu\nu,\alpha} - \Gamma^\beta_{\alpha\mu} g_{\beta
u} - \Gamma^\beta_{\alpha\nu} g_{\beta\mu}. $$

Define $\Gamma_{\alpha\mu\nu} = g_{\beta\nu} \Gamma^\beta_{\alpha\mu}$. Note, this $\Gamma_{\alpha\mu\nu}$ is the 1-form index.

Also, in an orthonormal vielbein, $g_{\mu\nu} = \eta_{\mu\nu}$ so $g_{\mu\nu,\alpha} = 0$. Thus,

$$\Gamma_{\mu\nu,\alpha} + \Gamma_{\nu\alpha\mu} = 0,$$

and $\Gamma_{\mu\nu,\alpha}$ antisymmetric in $\mu, \nu$. (Recall in coord. basis w. LC connection, $\Gamma_{\mu\nu} = \Gamma_{\nu\mu}$) & This property depends only on $\nabla g = 0$ (the parallel transport proceed by orthogonal (or Lorentz) transformation), it does not require the LC connection.

In terms of Cartan's forms, this condition is

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \quad (\omega_{\mu\nu} = \frac{\eta_{\mu\alpha}}{\sqrt{g}} \omega^{\alpha\nu}).$$

Similarly, we have

$$R_{\mu\nu} = -R_{\nu\mu} \quad (R_{\mu\nu} = \frac{\eta_{\mu\alpha}}{\sqrt{g}} R_{\alpha\nu}, \text{ Riemann-Cartan})$$

for the same reason. Also note, if in addition $T=0$ (LC connection) then

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} (c^\alpha_{\mu\nu} + c^\alpha_{\nu\mu} + c^\alpha_{\nu\mu}).$$

Now we consider a change of basis for an orthonormal vielbein.

To be specific, we'll assume the pseudo-Riemannian (1+3) case, with $g_{\mu\nu} = \eta_{\mu\nu}$. A change of basis maps one orthonormal vielbein to another.

We are assuming that $g(e_\alpha, e_\beta) = \eta_{\alpha\beta}$.

Let $e^{\mu}_{\alpha} = \Lambda^{\alpha}_{\mu} e_{\alpha}$, and demand that $g(e^{\mu}_{\alpha}, e^{\nu}_{\beta}) = \eta_{\mu\nu},$ so the new vielbein is also orthonormal.
Let $e'_{\alpha} = \Lambda_{\alpha}^{\beta} e_{\beta}$, defines $\Lambda_{\alpha}^{\beta}$. Then demand

\[ g(e'_{\alpha}, e'_{\beta}) = \eta_{\alpha\beta}, \]

and you find

\[ \Lambda^{\alpha}_{\mu} \Lambda^{\beta}_{\nu} \eta_{\alpha\beta} = \eta_{\mu\nu} \]

where indices are raised + lowered with $\eta$. Thus $\Lambda^{\alpha}_{\mu}(x)$ is an $x$-dependent Lorentz transformation. These are gauge transformations in GR. How other things transform:

\[ \theta'^{\mu} = \Lambda^{\mu}_{\alpha} \theta^{\alpha}. \]

Any tensor transforms pointwise-linearly in $\Lambda(x)$, for example, the Riemann-Cartan 2-form,

\[ R'^{\mu}_{\nu} = \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} R^{\alpha}_{\beta}. \]

But the Cartan-Connection 1-form has a less simple transformation law (since $\Gamma$ is not a tensor):

\[ \omega'^{\gamma}_{\nu} = \Lambda^{\gamma}_{\alpha} \Lambda^{\nu}_{\beta} \omega^{\alpha}_{\beta} - \Lambda^{\gamma}_{\alpha} \Lambda^{\nu}_{\beta} (\Lambda^{-1})^{\alpha}_{\gamma} \theta^{\alpha}. \]

The extra term on the right is characteristic of the transformation laws for gauge potentials.
Now we deal with the variational formulation of GR. We work in coordinates $x^k$. We start with the vacuum (matter-free) case, for which the field eqs are

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0.$$ 

We seek a Lagrangian density $L_g$ such that these eqs follow from

$$\delta \int d^4x \sqrt{-g} L_g = 0.$$ 

Here $d^4x = dx^\alpha dx^\beta dx^\gamma dx^\delta$, $g = \text{det } g_{\mu\nu} < 0$, so $-g = |g|$. The product $d^4x \sqrt{-g}$ is the invariant volume element, as will be explained later in the course. $L_g$ must be a scalar in order that the integral be independent of coordinates. The simplest scalar that can be constructed out of $g_{\mu\nu}$ and its derivatives (apart from trivial things like $g^{\mu\nu} = 2$) is the curvature scalar $R$. So we guess that $L_g \propto R$, and we look at the variation,

$$\delta \int d^4x \sqrt{-g} R = 0.$$ 

The variation is carried out by $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$. First we compute the variation in $g^{\mu\nu}$ induced by $\delta g_{\mu\nu}$. Use

$$g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}_{\rho} \Rightarrow$$

$$\delta g^{\mu\nu} g_{\nu\rho} + g^{\mu\nu} \delta g_{\nu\rho} = 0$$

$$\Rightarrow \delta g^{\mu\nu} = -g^{\mu\sigma} g^{\nu\beta} \delta g_{\sigma\beta}$$
Next we compute $8 \sqrt{-g}$. Let $M$ be a matrix that depends on a parameter $\lambda$. Then we have the useful identity,

$$\frac{d}{d\lambda} (\det M) = (\det M) \text{tr} \left( M^{-1} \frac{dM}{d\lambda} \right).$$

Identify $M$ with $g_{\mu\nu}$, $\det M = g$, this implies

$$\delta g = g (g^{\mu\nu} \delta g_{\mu\nu}),$$

or

$$8 \sqrt{-g} = \frac{1}{2} \sqrt{-g} \left( g^{\mu\nu} \delta g_{\mu\nu} \right).$$

Finally, we need $8R$. Start with $8 \Gamma^\mu_{\alpha\beta}$, the change in the L.C. $\Gamma$ when $g_{\mu\nu}$ goes to $g_{\mu\nu} + \delta g_{\mu\nu}$. Being the change difference between 2 connections, this is a tensor, which we will write as $(8\Gamma)^{\nu}_{\alpha\beta}$. Be careful about the positions of the indices. Of course $\Gamma^\mu_{\alpha\beta}$ itself is not a tensor.

Now we compute $8R^{\mu}_{\nu\alpha\beta}$ in terms of $8\Gamma$. The expression for $R$ has the structure

$$R = 2\Gamma + \Gamma + \Gamma - \Gamma,$$

omitting all indices. Therefore

$$8R = 2(8\Gamma) - 2(8\Gamma) + (8\Gamma) + (8\Gamma) - (8\Gamma) - (8\Gamma).$$

We evaluate $8R^{\mu}_{\nu\alpha\beta}$ at an arbitrary point of the manifold that we call $0$, $8R^{\mu}_{\nu\alpha\beta}(0)$. We use Riemann normal coordinates based at $0$, so $\Gamma^\mu_{\nu\alpha}(0) = 0$. Thus

$$8R^{\mu}_{\nu\alpha\beta}(0) = (8\Gamma)^{\nu}_{\alpha\beta}(0) - (8\Gamma)^{\nu}_{\alpha\beta}(0),$$

since all terms in $\Gamma - 8\Gamma$ vanish. Since $8\Gamma$ is a tensor, both terms above are ordinary derivatives of tensors, evaluated at $0$. 
But in R.N.C., such odd derivs are equal to covariant derivs, (evaluated at 0). So we can replace the comma with a semicolon.

Then we have a relation between two tensors,

\[ \delta R^\mu_{\nu \rho} (0) = (\delta \Gamma)^{\mu}_{\nu \rho \alpha} (0) - (\delta \Gamma)^{\nu}_{\mu \alpha \rho} (0). \]

But since 0 was arbitrary, this is true at all points,

\[ \delta R^\mu_{\nu \rho} = (\delta \Gamma)^{\mu}_{\nu \rho \alpha} - (\delta \Gamma)^{\nu}_{\mu \alpha \rho}. \]

And since it is a tensor eqn, it is valid in all coordinates (not only RNC).

Now by contracting, we get the variation of the Rieci tensor,

\[ \delta R_{\nu \rho} = (\delta \Gamma)^{\nu}_{\rho \mu \alpha} - (\delta \Gamma)^{\rho}_{\nu \alpha \mu}. \]

or juggling indices,

\[ \delta R_{\mu \nu} = (\delta \Gamma)^{\alpha}_{\mu \nu \alpha} - (\delta \Gamma)^{\alpha}_{\nu \mu \alpha}. \]

Finally, as for the curvature scalar, we have \( R = g^{\mu \nu} R_{\mu \nu} = R^\mu_{\mu} \),

\[ \delta R = g^{\mu \nu} \delta R_{\mu \nu} + g^{\mu \nu} \delta R_{\mu \nu} 
\]

\[ = -g^{\mu \nu} g^{\rho \beta} \delta g_{\rho \beta} R_{\mu \nu} + g^{\mu \nu} \left( (\delta \Gamma)^{\nu}_{\rho \mu \alpha} - (\delta \Gamma)^{\rho}_{\nu \alpha \mu} \right) \]

\[ \delta R = -R^{\mu \nu} \delta g_{\mu \nu} + (\delta \Gamma)^{\mu \nu}_{\rho \mu \alpha} - (\delta \Gamma)^{\nu \mu}_{\rho \alpha \mu}. \]

Thus,

\[ \delta \int \! d^{4}x \sqrt{-g} \ R = \int \! d^{4}x \left[ \delta \sqrt{-g} \ R + \sqrt{-g} \ \delta R \right] \]

\[ = \int \! d^{4}x \sqrt{-g} \left[ \frac{1}{2} g^{\mu \nu} \delta g_{\mu \nu} + R^{\mu \nu} \delta g_{\mu \nu} + (\delta \Gamma)^{\mu \nu}_{\rho \mu \alpha} - (\delta \Gamma)^{\nu \mu}_{\rho \alpha \mu} \right] \]

\[ = 4 \ \text{terms}. \]
The last two terms vanish on integration. For example, let

\[ x^\alpha = 8\alpha^{\mu}\mu, \]

so the expression

\[ x^\alpha_{;\alpha} \]

(the covariant divergence of a vector) appears in the integral. This can also be written,

\[ x^\alpha_{;\alpha} = \frac{1}{\sqrt{-g}} (\nabla^\alpha x^\alpha),\alpha \]

by an identity we will prove shortly. Thus

\[ \int d^4x \sqrt{-g} x^\alpha_{;\alpha} = \int d^4x (\sqrt{-g} x^\nu),\alpha = 0 \]

by integration by parts (X vanishes at \( \infty \)). (Or maybe \( M \) is compact). Similarly for the 4th term. Thus,

\[ \delta \int_\Sigma \sqrt{-g} d^4x \ R = \int d^4x \sqrt{-g} \left[ +\frac{1}{2} g^{\mu\nu} R - R^{\mu\nu} \right] \delta g_{\mu\nu} = 0 \]

for all \( \delta g_{\mu\nu} \Rightarrow \)

\[ +\frac{1}{2} g^{\mu\nu} R - R^{\mu\nu} = - G^{\mu\nu} = 0 \]

the vacuum Einstein equations.

Conventionally we take

\[ L_G = \frac{R}{16\pi G} \]

\( G = \) Newton's constant of gravitation, henceforth set to 1.

If a matter Lagrangian \( L_m \) is added to \( L_G \) and the overall variational principle is
\[ \delta \int d^4x \sqrt{-g} \left( \mathcal{L}_G + \mathcal{L}_M \right) = 0 \]

with \( \mathcal{L}_G = R/16\pi \), then to get the right field equations,

\[ G_{\mu \nu} = 8\pi T_{\mu \nu} \]

we must have

\[ \frac{\delta}{\delta g_{\mu \nu}} \int d^4x \sqrt{-g} \mathcal{L}_M = \frac{1}{2} T^{\mu \nu}. \]

Now we turn to the problem of putting spinors into curved space-time. The idea is to add the Dirac Lagrangian to the gravitational on \( \mathcal{L}_G \).

In special relativity (SR), the Dirac Lagrangian is

\[ \mathcal{L}_D = \bar{\Psi} \left( i \gamma^\mu \partial_\mu - m \right) \Psi \]

in units \( \hbar = c = 1 \). Notation is standard, \( \gamma^\mu \) are the Dirac 4x4 matrices, \( \Psi \) is the Dirac 4-spinor, and \( \partial_\mu \) means differentiation w.r.t. flat space coordinates \( (t, \vec{x}) = x^\mu \).

There are two problems on putting this into GR. The first is that the usual \( \gamma^\mu \) matrices are tied to inertial frames in SR, i.e. coordinates \( x^\mu = (t, \vec{x}) \). Rather than trying to generalize the \( \gamma^\mu \) matrices to other frames, a better choice is to introduce an orthonormal vierbein \( e^\mu_\nu \),

\[ g_{\mu \nu} = \eta_{\mu \nu}, \text{ and replace } \partial_\mu \text{ by } e_\mu. \]

Then we can use the standard \( \gamma^\mu \) matrices of SR even in GR.

The second problem is that \( \mathcal{L}_D \) is not covariant, so \( \mathcal{L}_D \) is not a scalar \( \phi \) in GR, as written. Obviously we must replace \( \mathcal{L}_D \).
with $\nabla \Psi$, where $\nabla = \nabla_a$ is a covariant derivative. But how do we compute covariant derivatives of spinors?

Take our clue from the covariant derivative of ordinary vectors. Begin with parallel transport of vector $Y \in T_x M$ to $Y' \in T_{x+\Delta x} M$,

\[
\begin{array}{c}
Y \\
\downarrow \Delta x \\
Y' \\
\Delta x \\
x+\Delta x
\end{array}
\]

In some local chart $x^\mu$, we know that

\[ Y'^\mu = (\delta^\mu_\nu - \Delta x^\sigma \Gamma^\mu_{\nu \sigma}) Y^\nu \]

where $\Gamma^\mu_{\nu \sigma}$ are the connection coefficients w.r.t. the chart $x^\mu$. If we transform this to an ON vierbein $e^a \xi$, then we have

\[ Y'^a = (\delta^a_\beta - \Delta x^\gamma \Gamma^a_{\gamma \beta}) Y^\beta \]

where now $\Gamma^a_{\gamma \beta}$ is the connection coefficients w.r.t. to the vierbein. It is an equation of the same form, in spite of the fact that $\Gamma$ does not transform as a tensor. Now, however, the matrix

\[ \Delta^a_{\beta} = (\mathcal{I} + \Omega)^a_{\beta} = \delta^a_\beta - \Delta x^\gamma \Gamma^a_{\gamma \beta} \]

is an infinitesimal Lorentz transformation, where the correction term

\[ \Omega^a_{\beta} = -\Delta x^\gamma \Gamma^a_{\gamma \beta} \]

satisfies $\Omega_{\alpha \beta} = -\Omega_{\beta \alpha}$ (it is an element of the Lie algebra of $\mathfrak{so}(3,1)$.)
To parallel transport Dirac spinors from $x$ to $x + \Delta x$, say,

$$\psi \rightarrow \psi'$$

we may apply the Lorentz transformation $D(\Lambda)$ to $\psi$, where $\Lambda = 1 + \Sigma$

is the infinitesimal Lorentz transformation defined above. Here $D(\Lambda)$ is the representation of the Lorentz group for Dirac spinors. Actually, it is not a representation, since it is double-valued. More about that next week.
Summary:

Vectors, infinitesimal transport:

\[
Y'_{\mu} = (\delta_{\nu}^{\mu} - \Delta x^\alpha \Gamma^\mu_{\nu \alpha}) Y^\nu \quad \text{(tov)}
\]

\[
Y'_{\alpha} = (\delta_{\beta}^{\alpha} - \Delta x^\gamma \Gamma^\alpha_{\beta \gamma}) Y^\beta \quad \text{(O.N. vielbein)}
\]

\[
\Delta = (I + \Omega) \gamma^\alpha, \quad \Omega \epsilon O(3,1)
\]

\[
\Omega_{\alpha \beta} = -\Delta x^\gamma \Gamma^\gamma_{\alpha \beta} = -\Omega_{\alpha \beta}
\]

Equations:

\[
\psi' = \mathcal{D}(\Delta) \psi
\]

\[\mathcal{D}(\Delta) = \text{Diag. } "\text{representation}\" \text{ of Lorentz group. Actually, it is not a representation (it's double valued), and it's not a rep. of the whole Lorentz group, only the proper, orthochronous Lorentz group. More on all that in a moment, for now be sloppy and just write } \mathcal{D}(\Delta), \text{ and call on standard material on Lorentz transforming the Dirac equation. This tells us, for infinitesimal Lorentz transformations, (} \Omega \ll 1)\]

\[
\mathcal{D}(I + \Omega) = 1 + \frac{i}{4} \Omega_{\alpha \beta} \sigma^{\alpha \beta},
\]

where \( \sigma^{\alpha \beta} = \frac{i}{2} [\gamma^\alpha, \gamma^\beta] \) \text{ (standard notation for Dirac matrices). Recall } \{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha \beta}
Summary of the relevant facts regarding Dirac matrices and Lorentz transforming Dirac spinors:

Lorentz transforming Dirac spinors:

\[ D(\Lambda_1)D(\Lambda_2) = \pm D(\Lambda_1\Lambda_2) \]

\[ \Psi' = D(\Lambda) \Psi \]  
(Lorentz transforming Dirac spinor)

\[ D(\Lambda)^{-1} \gamma^\alpha D(\Lambda) = \Lambda^\alpha_\beta \gamma^\beta \]  
(\( \gamma^\alpha \) transforms as a 4-vector)

\[ \gamma^0 D(\Lambda)^+ \gamma^0 = D(\Lambda)^{-1} \]

Hence

\[ \Psi' = \left( 1 + \frac{i}{4} \gamma^\nu \Gamma_{\lambda\nu\beta} \gamma^\beta \right) \Psi \]

So, basic idea is that under an infinitesimal parallel transport, a spinor transforms by the same (infinitesimal) Lorentz transformation as a vector, but the spinor rep. of the L.T. must be used.

Now, the covariant derivative is defined by the parallel transport. We put \( \Delta x^\mu = \varepsilon \chi^\mu \) where \( \chi \in T_x M \), and define (for \( Y \in \mathfrak{X}(M) \))

\[ \nabla_x Y = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ Y(x + \varepsilon X) - Y \right] \]

gives \( (\nabla_x Y)^\mu = x^\nu \left( Y^\mu_{,\nu} + \Gamma^\mu_{\nu\sigma} Y^\sigma \right) \).

Similarly, define (for a spinor field \( \psi(x) \)):

\[ \nabla_x \psi = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \psi(x + \varepsilon X) - \psi \right] \]

\[ \nabla_\alpha \psi = x^\alpha \left[ \psi_{,\alpha} - \frac{i}{4} \Gamma_{\beta\gamma} \bar{\psi} \gamma^\beta \gamma^\gamma \psi \right] \]

\[ \nabla_{\bar{\alpha}} \psi = \psi_{,\bar{\alpha}} - \frac{i}{4} \Gamma_{\bar{\beta}\bar{\gamma}} \bar{\psi} \gamma^\beta \gamma^\gamma \psi \]

Covariant derivative of Dirac spinors.