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$$= \frac{1}{2}\left(\Gamma^{\mu}_{\beta\nu,\alpha} - \Gamma^{\mu}_{\alpha\nu,\beta} - \Gamma^{\mu}_{\sigma\nu} c^{\sigma}_{\alpha\beta} \right) \theta^{\alpha} \wedge \theta^{\beta}$$

$$= \frac{1}{2}\left(R^{\mu}_{\cdot\nu\alpha\beta} - \Gamma^{\sigma}_{\beta\nu}\Gamma^{\mu}_{\alpha\sigma} + \Gamma^{\sigma}_{\alpha\nu}\Gamma^{\mu}_{\beta\sigma} \right) \theta^{\alpha} \wedge \theta^{\beta}$$

$$= R^{\mu}_{\cdot\nu} - \frac{1}{2}\, \omega^{\mu}_{\cdot\sigma} \wedge \omega^{\sigma}_{\cdot\nu} + \frac{1}{2}\, \omega^{\sigma}_{\cdot\nu} \wedge \omega^{\mu}_{\cdot\sigma},$$

equal

or,

$$\boxed{d\omega^{\mu}_{\cdot\nu} + \omega^{\mu}_{\cdot\sigma} \wedge \omega^{\sigma}_{\cdot\nu} = R^{\mu}_{\cdot\nu}}$$

2nd Cartan structure eqn.

Again, take

$$T^{\mu} = d\theta^{\mu} + \omega^{\mu}_{\cdot\alpha} \wedge \theta^{\alpha}, \qquad \text{apply } d,$$

$$dT^{\mu} = 0 + \left(R^{\mu}_{\cdot\alpha} - \cancel{\omega^{\mu}_{\cdot\sigma} \wedge \omega^{\sigma}_{\cdot\alpha}} \right) \wedge \theta^{\alpha}$$

$$- \omega^{\mu}_{\cdot\alpha} \wedge d\theta^{\alpha}$$

$$d\theta^{\alpha} \to T^{\alpha} - \cancel{\omega^{\alpha}_{\cdot\beta} \wedge \theta^{\beta}}$$

$$\boxed{dT^{\mu} + \omega^{\mu}_{\cdot\alpha} \wedge T^{\alpha} = R^{\mu}_{\cdot\alpha} \wedge \theta^{\alpha}}$$

1st Bianchi identity, generalized to case $T \neq 0$.

Finally, take 2nd Cartan structure, apply d:

$$d R^{\mu}_{\cdot\nu} = d\omega^{\mu}_{\cdot\sigma} \wedge \omega^{\sigma}_{\cdot\nu} - \omega^{\mu}_{\cdot\sigma} \wedge d\omega^{\sigma}_{\cdot\nu}$$

$$= \left(R^{\mu}_{\cdot\sigma} - \cancel{\omega^{\mu}_{\cdot\alpha} \wedge \omega^{\alpha}_{\cdot\sigma}} \right) \wedge \omega^{\sigma}_{\cdot\nu}$$

$$- \omega^{\mu}_{\cdot\sigma} \wedge \left(R^{\sigma}_{\cdot\nu} - \cancel{\omega^{\sigma}_{\cdot\alpha} \wedge \omega^{\alpha}_{\cdot\nu}} \right)$$

one might say that the covariant exterior derivative of the curvature 2-form is 0, that this form is closed in this sense.

$$\boxed{dR^{\mu}{}_{\cdot\nu} + \omega^{\mu}{}_{\cdot\sigma}\wedge R^{\sigma}{}_{\cdot\nu} - R^{\mu}{}_{\cdot\sigma}\wedge\omega^{\sigma}{}_{\cdot\nu} = 0}$$

2nd Bianchi, generalized.

When $T=0$, these eqns should reduce to the previous versions of the Bianchi identities. For the 1st Bianchi ident., this gives

$$0 = R^{\mu}{}_{\cdot\alpha}\wedge\theta^{\alpha} = \frac{1}{2} R^{\mu}{}_{\cdot\nu\alpha\beta}\ \theta^{\nu}\wedge\theta^{\alpha}\wedge\theta^{\beta}$$

$$\Rightarrow R^{\mu}{}_{\cdot[\nu\alpha\beta]} = 0. \qquad \text{checks.}$$

For the 2nd Bianchi ident., notice that it doesn't involve T at all. But if you want to show equivalence to $R^{\mu}{}_{\cdot\nu[\alpha\beta;\gamma]} = 0$, you must use $T=0$.

Now consider the case that we have a metric g and a metric connection $\nabla g = 0$.

Then it is convenient to assume the basis $\{e_\alpha\}$ is <u>orthonormal</u>, i.e.,

$$g_{\alpha\beta} = g(e_\alpha, e_\beta) = \eta_{\alpha\beta} \qquad (\text{pseudo-Riem. case, or } \delta_{\alpha\beta}, \text{ Riem. case}).$$

$$= \text{const. metric of special relativity.}$$

We know that if the curvature tensor $\neq 0$, then there is no <u>coordinate basis</u> such that $g_{\alpha\beta} = \eta_{\alpha\beta}$. But there are always <u>non-coordinate</u> bases that make this true. This is a special kind of vielbein.

There are some special properties of Γ, R in orthonormal vielbeins. First, $\nabla g = 0$ implies 11/6/08

$$0 = g_{\mu\nu,\alpha} - \Gamma^{\beta}_{\alpha\mu} g_{\beta\nu} - \Gamma^{\beta}_{\alpha\nu} g_{\mu\beta}.$$

Define $\Gamma_{\alpha\mu\nu} = g_{\alpha\beta} \Gamma^{\beta}_{\mu\nu}$. Note, this $\Gamma_{\alpha\mu\nu}$ is the 1-form index.

Also, in an orthonormal vielbein, $g_{\mu\nu} = \eta_{\mu\nu}$ so $g_{\mu\nu,\alpha} = 0$. Thus,

$$\Gamma_{\mu\alpha\nu} + \Gamma_{\nu\alpha\mu} = 0,$$

and $\Gamma_{\mu\alpha\nu}$ is antisymmetric in $\mu\nu$. (Recall in coord. basis w. LC connection, $\Gamma_{\alpha\mu\nu} = \Gamma_{\alpha\nu\mu}$.) This property depends only on $\nabla g = 0$ (the parallel transport proceed by orthogonal (or Lorentz) transformations), it does not require the LC connection.

In terms of Cartan's forms, this condition is

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \qquad \left(\omega_{\mu\nu} = \eta_{\mu\alpha} \omega^{\alpha}_{\cdot\nu} \right).$$

Similarly, we have

$$R_{\mu\nu} = -R_{\nu\mu} \qquad \left(R_{\mu\nu} = \eta_{\mu\alpha} R^{\alpha}_{\cdot\nu}, \text{ Riemann-Cartan tensor} \right)$$

for the same reason.

Also note, if in addition $T=0$ (L.C. connection) then $\Gamma^{\alpha}_{\mu\nu} = +\frac{1}{2}\left(C^{\alpha}{}_{\mu\nu} + C_{\mu}{}^{\alpha}{}_{\nu} + C_{\nu}{}^{\alpha}{}_{\mu} \right)$

Now we consider a change of basis for an orthonormal vielbein. To be specific, we'll assume the pseudo-Riemannian (1+3) case, with $g_{\mu\nu} = \eta_{\mu\nu}$. A change of basis maps one orthonormal vielbein to another.

We are assuming that $g(e_\alpha, e_\beta) = \eta_{\alpha\beta}$.

Let $e'_\mu = \Lambda^{\alpha}_{\cdot\mu} e_\alpha$, and demand that $g(e'_\mu, e'_\nu) = \eta_{\mu\nu}$, so the new vielbein is also orthonormal.

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Let $e'_\alpha = \Lambda_\alpha{}^\beta e_\beta$, defines $\Lambda_\alpha{}^\beta$. Then demand $g(e'_\alpha, e'_\beta) = \eta_{\alpha\beta}$, and you find

$$\Lambda^\alpha{}_{\cdot\mu}\, \eta_{\alpha\beta}\, \Lambda^\beta{}_{\cdot\nu} = \eta_{\mu\nu}$$

where indices are raised + lowered with η. Thus $\Lambda^\alpha{}_{\cdot\mu}(x)$ is an <u>x-dependent Lorentz transformation</u>. <u>These are gauge transformations in GR.</u> How other things transform:

$$\theta'^\mu = \Lambda^\mu{}_{\cdot\alpha}\, \theta^\alpha .$$

Any tensor transforms pointwise-linearly in $\Lambda(x)$, for example, the Riemann-Cartan 2-form,

$$R'^\mu{}_{\cdot\nu} = \Lambda^\mu{}_{\cdot\alpha}\, \Lambda_\nu{}^\beta\, R^\alpha{}_{\cdot\beta} .$$

But the Cartan-Connection 1-form has a less simple transformation law (since Γ is not a tensor):

$$\omega'^\sigma{}_{\cdot\nu} = \Lambda^\sigma{}_{\cdot\gamma}\, \Lambda_\nu{}^\beta\, \omega^\gamma{}_{\cdot\beta} - \Lambda^\sigma{}_{\cdot\gamma,\alpha}\, (\Lambda^{-1})^\gamma{}_{\cdot\nu}\, \theta^\alpha .$$

The extra term on the right is characteristic of the transformation laws for gauge potentials.

$$\boxed{\cancel{\omega'^\gamma{}_\beta = \Lambda_\sigma{}^\gamma\, \omega'^\sigma{}_{\cdot\nu}\, \Lambda^\nu{}_{\cdot\beta} + \Lambda_\sigma{}^\gamma\, \Lambda^\sigma{}_{\cdot\beta,\alpha}\, \theta^\alpha}}$$

Now we deal with the variational formulation of GR (in (3+1)-dim. space-time). We work in coordinates x^μ. We start with the vacuum (matter-free) case, for which the field eqns are

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 0.$$

We seek a Lagrangian density $\mathcal{L}_G$ such that these eqns follow from

$$\delta \int d^4x \sqrt{-g}\ \mathcal{L}_G = 0.$$

Here $d^4x = dx^0 \wedge \ldots \wedge dx^3$, $g = \det g_{\mu\nu} < 0$, so $-g = |g|$. The product $d^4x\sqrt{-g}$ is the <u>invariant</u> <u>volume</u> <u>element</u>, as will be explained later in the course. $\mathcal{L}_G$ must be a scalar in order that the integral be independent of coordinates. The simplest scalar that can be constructed out of $g_{\mu\nu}$ and its derivatives (apart from trivial things like $g^\mu{}_\mu = 4$) is the curvature scalar R. So we guess that $\mathcal{L}_G \propto R$, and we look at the variation,

$$\delta \int d^4x \sqrt{-g}\, R = 0.$$

The variation is carried out by $g_{\mu\nu} \mapsto g_{\mu\nu} + \delta g_{\mu\nu}$. First we compute the variation in $g^{\mu\nu}$ induced by $\delta g_{\mu\nu}$. Use

$$g^{\mu\alpha} g_{\alpha\beta} = \delta^\mu_\alpha \quad \Rightarrow$$

$$\delta g^{\mu\alpha} g_{\alpha\beta} + g^{\mu\alpha} \delta g_{\alpha\beta} = 0$$

$$\Rightarrow \quad \delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}$$

Next we compute $\delta\sqrt{-g}$. Let M be a matrix that depends on a parameter λ. Then we have the useful identity,

$$\frac{d}{d\lambda}(\det M) = (\det M)\,\mathrm{tr}\left(M^{-1}\frac{dM}{d\lambda}\right).$$

Identify M with $g_{\mu\nu}$, $\det M = g$, this implies

$$\delta g = g\left(g^{\mu\nu}\delta g_{\mu\nu}\right),$$

or

$$\delta\sqrt{-g} = \tfrac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\delta g_{\mu\nu}\right).$$

Finally, we need δR. Start ~~by~~ with $\delta\Gamma^{\mu}_{\alpha\beta}$, the change in the L.C. Γ when $g_{\mu\nu}$ goes to $g_{\mu\nu}+\delta g_{\mu\nu}$. Being the ~~change~~ difference between 2 connections, this is a <u>tensor</u>, which we will write as $(\delta\Gamma)^{\mu}_{\cdot\alpha\beta}$ to be careful about the positions of the indices. Of course $\Gamma^{\mu}_{\alpha\beta}$ itself is not a tensor.

Now we compute $\delta R^{\mu}_{\cdot\nu\alpha\beta}$ in terms of $\delta\Gamma$. The expression for R has the structure

$$R = \partial\Gamma - \partial\Gamma + \Gamma\Gamma - \Gamma\Gamma,$$

omitting all indices. Therefore

$$\delta R = \partial(\delta\Gamma) - \partial(\delta\Gamma) + (\delta\Gamma)\Gamma + \Gamma(\delta\Gamma) - (\delta\Gamma)\Gamma - \Gamma(\delta\Gamma).$$

We evaluate $\delta R^{\mu}_{\cdot\nu\alpha\beta}$ at an arbitrary point of the manifold that we call O, $\delta R^{\mu}_{\cdot\nu\alpha\beta}(O)$. We use Riemann normal coordinates based at O, so $\Gamma^{\mu}_{\nu\alpha}(O) = 0$. Thus

$$\delta R^{\mu}_{\cdot\nu\alpha\beta}(O) = (\delta\Gamma)^{\mu}_{\cdot\beta\nu,\alpha}(O) - (\delta\Gamma)^{\mu}_{\cdot\alpha\nu,\beta}(O),$$

since all 4 terms in $\Gamma\cdot\delta\Gamma$ vanish. Since $\delta\Gamma$ is a tensor, ~~all~~ both terms above are ordinary derivatives of tensors, evaluated at O.

But in R.N.C., such ord. derivs are equal to covariant derivs, 11/6/08
(evaluated at 0). So we can replace the comma with a semicolon.
Then we have a relation between two tensors,

$$\delta R^{\mu}{}_{\nu\alpha\beta}(0) = (\delta\Gamma)^{\mu}{}_{\beta\nu;\alpha}(0) - (\delta\Gamma)^{\mu}{}_{\alpha\nu;\beta}(0).$$

But since 0 was arbitrary, this is true at all points,

$$\delta R^{\mu}{}_{\nu\alpha\beta} = (\delta\Gamma)^{\mu}{}_{\beta\nu;\alpha} - (\delta\Gamma)^{\mu}{}_{\alpha\nu;\beta}$$

And since it is a tensor eqn, it is valid in all coordinates (not only RNC).

Now by contracting, we get the variation of the Ricci tensor,

$$\delta R_{\nu\beta} = (\delta\Gamma)^{\alpha}{}_{\beta\nu;\alpha} - (\delta\Gamma)^{\alpha}{}_{\alpha\nu;\beta}$$

or juggling indices

$$\delta R_{\mu\nu} = (\delta\Gamma)^{\alpha}{}_{\nu\mu;\alpha} - (\delta\Gamma)^{\alpha}{}_{\alpha\mu;\nu}.$$

Finally, as for the curvature scalar, we have $R = g^{\mu\nu}R_{\mu\nu} = R_{\mu}{}^{\mu}$,

$$\delta R = \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu}\,\delta R_{\mu\nu}$$

$$= -g^{\mu\alpha}g^{\nu\beta}\,\delta g_{\alpha\beta} R_{\mu\nu} + g^{\mu\nu}\left((\delta\Gamma)^{\alpha}{}_{\nu\mu;\alpha} - (\delta\Gamma)^{\alpha}{}_{\alpha\mu;\nu}\right)$$

$$\delta R = -R^{\mu\nu}\,\delta g_{\mu\nu} + (\delta\Gamma)^{\alpha\mu}{}_{\mu;\alpha} - (\delta\Gamma)^{\alpha}{}_{\alpha}{}^{\mu}{}_{;\mu}.$$

Thus,

$$\delta\int d^4x\sqrt{-g}\,R = \int d^4x\left[\delta\sqrt{-g}\,R + \sqrt{-g}\,\delta R\right]$$

$$= \int d^4x\sqrt{-g}\left[\tfrac{1}{2}g^{\mu\nu}R\,\delta g_{\mu\nu} - R^{\mu\nu}\delta g_{\mu\nu} + (\delta\Gamma)^{\alpha\mu}{}_{\mu;\alpha} - (\delta\Gamma)^{\alpha}{}_{\alpha}{}^{\mu}{}_{;\mu}\right]$$

= 4 terms.

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The last two terms vanish on integration. For example, let

$$X^\alpha = \delta\Gamma^{\alpha\mu}{}_{;\mu},$$

so the expression

$$X^\alpha{}_{;\alpha}$$

(the covariant divergence of a vector) appears in the integral. This can also be written,

$$X^\alpha{}_{;\alpha} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,X^\alpha\right)_{,\alpha}$$

by an identity we will prove shortly. Thus

$$\int d^4x\sqrt{-g}\;X^\alpha{}_{;\alpha} = \int d^4x\;\left(\sqrt{-g}\,X^\alpha\right)_{,\alpha} = 0$$

by integration by parts (X vanishes at ∞). (or maybe M is compact). Similarly for the 4th term. Thus,

$$\delta\int\sqrt{-g}\,d^4x\;R = \int d^4x\sqrt{-g}\left[+\tfrac{1}{2}g^{\mu\nu}R - R^{\mu\nu}\right]\delta g_{\mu\nu} = 0$$

for all $\delta g_{\mu\nu}$ $\Rightarrow$

$$+\tfrac{1}{2}g^{\mu\nu}R - R^{\mu\nu} = -G^{\mu\nu} = 0$$

the vacuum Einstein equations.

Conventionally we take

$$\mathcal{L}_G = \frac{R}{16\pi G},$$

G = Newton's constant of gravitation, henceforth set to 1.

If a matter Lagrangian $\mathcal{L}_M$ is added to $\mathcal{L}_G$ and the overall variational principle is

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$$\delta \int d^4x \sqrt{-g}\,(\mathcal{L}_G + \mathcal{L}_M) = 0$$

with $\mathcal{L}_G = R/16\pi$, then to get the right field equations,

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

we must have

$$\frac{\delta}{\delta g_{\mu\nu}} \int d^4x \sqrt{-g}\, \mathcal{L}_M = \frac{1}{2} T^{\mu\nu}.$$

Now we turn to the problem of putting spinors into curved space-time. The idea is to add the Dirac Lagrangian to the gravitational on $\mathcal{L}_G$. In special relativity, (SR), the Dirac Lagrangian is

$$\mathcal{L}_D = \bar{\psi}\,(i\gamma^\alpha \partial_\alpha - m)\,\psi$$

in units $\hbar = c = 1$. Notation is standard, γ^α are the Dirac 4×4 matrices, ψ is the Dirac 4-spinor, and ∂_α means differentiation wrt. flat space coordinates $(t, \vec{x}) = x^\mu$.

There are two problems on putting this into GR. The first is that the usual γ matrices are tied to inertial frames in SR, i.e. coordinates $x^\mu = (t, \vec{x})$. Rather than trying to generalize the γ matrices to other frames, a better choice is to introduce an orthonormal vierbein $\{e_\mu\}$, $g_{\mu\nu} = \eta_{\mu\nu}$, and replace ∂_α by e_α. Then we can use the standard γ matrices of SR even in GR.

The second problem is that $e_\alpha \psi$ ($= \psi_{,\alpha}$ in our generalized comma notation) is not covariant, so $\mathcal{L}_D$ is not a scalar in GR, as written. Obviously we must replace $e_\alpha \psi$

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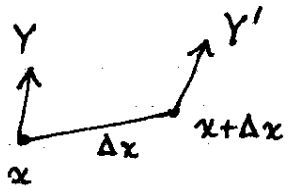
with $\nabla_\alpha \psi$, where $\nabla_\alpha \equiv \nabla_{e_\alpha}$ is a covariant derivative. But how do we compute covariant derivatives of spinors?

Take our clue from the covariant derivative of ordinary vectors. Begin with <u>parallel transport</u> of vector $Y \in T_x M$ to $Y' \in T_{x+\Delta x} M$,



In some local chart x^μ, we know that

$$Y'^\mu = \left(\delta^\mu_\nu - \Delta x^\sigma \Gamma^\mu_{\sigma\nu}\right) Y^\nu$$

where $\Gamma^\mu_{\sigma\nu}$ are the connection coefficients w.r.t. the chart x^μ. If we transform this to an ON vierbein $\{e_\alpha\}$, then we have

$$Y'^\alpha = \left(\delta^\alpha_\beta - \Delta x^\gamma \Gamma^\alpha_{\gamma\beta}\right) Y^\beta,$$

where now $\Gamma^\alpha_{\gamma\beta}$ is the connec. coeffs. w.r.t. to the vierbein. It is an equation of the same form, in spite of the fact that Γ does not transform as a tensor. Now, however, the matrix

$$\Lambda^\alpha{}_\beta = (I + \Omega)^\alpha{}_\beta = \delta^\alpha_\beta - \Delta x^\gamma \Gamma^\alpha_{\gamma\beta}$$

is an infinitesimal Lorentz transformation, where the correction term

$$\Omega_{\alpha\beta} = -\Delta x^\gamma \Gamma_{\alpha\gamma\beta}$$

satisfies $\Omega_{\alpha\beta} = -\Omega_{\beta\alpha}$ (it is an element of the Lie algebra of $O(3,1)$.)

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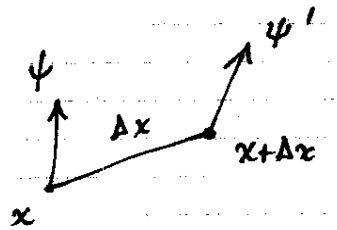
To parallel transport Dirac spinors from x to $x+\Delta x$, say,

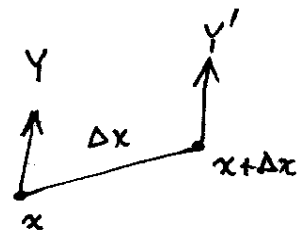
$\psi \mapsto \psi'$



we may apply the Lorentz transformation $D(\Lambda)$ to ψ, where $\Lambda = I + \Omega$ is the infinitesimal Lorentz transformation defined above. Here $D(\Lambda)$ is the representation of the Lorentz group for Dirac spinors. Actually, it is not a representation, since it is double-valued. More about that next week.

Summary.

Vectors, infinitesimal ∥ transport:



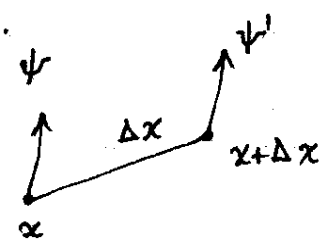
$$Y'^{\mu} = \left(\delta^{\mu}_{\nu} - \Delta x^{\sigma}\,\Gamma^{\mu}_{\sigma\nu}\right) Y^{\nu} \quad \text{(coord)}$$

$$Y'^{\alpha} = \underbrace{\left(\delta^{\alpha}_{\beta} - \Delta x^{\gamma}\,\Gamma^{\alpha}_{\gamma\beta}\right)}\, Y^{\beta} \quad \text{(O.N. vielbein)}$$

$$= (I + \Omega)^{\alpha}{}_{\beta}, \qquad \Lambda = I + \Omega \in O(3,1)$$

$$\Omega_{\alpha\beta} = -\Delta x^{\gamma}\,\Gamma_{\alpha\gamma\beta} = -\Omega_{\beta\alpha}.$$

Spinors:



$$\psi' = D(\Lambda)\psi$$

$D(\Lambda)$ = Dirac "representation" of Lorentz group. Actually, it's not a representation (it's double valued), and it's not a rep. of the whole Lorentz group, only the proper, orthochronous Lorentz group. More on all that in a moment, for now be sloppy and just write $D(\Lambda)$, and call on standard material on Lorentz transforming the Dirac equation. This tells us, for infinitesimal Lorentz transformations, $(\Omega \ll 1)$,

$$D(I + \Omega) = 1 - \frac{i}{4}\,\Omega_{\alpha\beta}\,\sigma^{\alpha\beta},$$

where $\sigma^{\alpha\beta} = \frac{i}{2}\left[\gamma^{\alpha}, \gamma^{\beta}\right]$ (standard notation for Dirac matrices). Recall $\{\gamma^{\alpha}, \gamma^{\beta}\} = 2\eta^{\alpha\beta}$

Summary of the relevant facts regarding Dirac matrices and Lorentz transforming Dirac spinors:

explain later.

$$D(\Lambda_1)D(\Lambda_2) = \pm D(\Lambda_1\Lambda_2)$$

$$\psi' = D(\Lambda)\psi \qquad \text{(Lorentz transforming Dirac spinor)}$$

$$D(\Lambda)^{-1}\gamma^\alpha D(\Lambda) = \Lambda^\alpha{}_\beta \gamma^\beta \qquad (\gamma^\alpha \text{ transforms as a 4-vector})$$

$$\gamma^0 D(\Lambda)^\dagger \gamma^0 = D(\Lambda)^{-1}$$

$$D(I+\Omega) = 1 - \frac{i}{4}\Omega_{\alpha\beta}\sigma^{\alpha\beta}.$$

Hence

$$\boxed{\psi' = \left(1 + \frac{i}{4}\Delta x^\gamma \Gamma_{\alpha\gamma\beta}\sigma^{\alpha\beta}\right)\psi}$$

So, basic idea is that under an infinitesimal parallel transport, a spinor transforms by the same (infinitesimal) Lorentz transformation as a vector, but the spinor rep. of the L.T. must be used.

Now, the covariant derivative is defined by the parallel transport. We put $\Delta x^\alpha = \varepsilon X^\alpha$ where $X \in T_x M$, and define (for $Y \in \mathcal{X}(M)$)

$$\nabla_X Y = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[Y(x+\varepsilon X) - Y'\right],$$

gives $(\nabla_X Y)^\mu = X^\nu\left(Y^\mu_{,\nu} + \Gamma^\mu_{\nu\sigma}Y^\sigma\right)$.

Similarly, define (for a spinor <u>field</u> $\psi(x)$):

$$\nabla_X\psi = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\psi(x+\varepsilon X) - \psi'\right]$$

$$\boxed{\begin{aligned}\nabla_X\psi &= X^\alpha\left[\psi_{,\alpha} - \frac{i}{4}\Gamma_{\beta\alpha\gamma}\sigma^{\beta\gamma}\psi\right]\\ \nabla_\alpha\psi &= \psi_{,\alpha} - \frac{i}{4}\Gamma_{\beta\alpha\gamma}\sigma^{\beta\gamma}\psi\end{aligned}}$$

Covariant derivs on Dirac spinors