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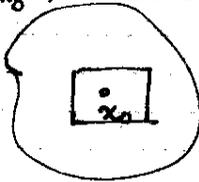
Shows appearance of Einstein tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

satisfies  $G^{\mu\nu}{}_{;\mu} = 0$ , necessary for field equations  $G_{\mu\nu} = 8\pi T_{\mu\nu}$

since  $T^{\mu\nu}{}_{;\mu} = 0$  (local energy-momentum conservation).

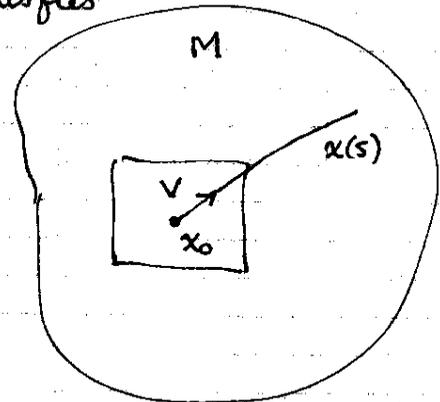
Now we consider Riemann normal coordinates. These are coordinates that simplify the expressions for tensors and covariant derivatives as much as possible in a neighborhood of a given point. We know that a small piece of  $M$  is approximately flat. Riemann normal coordinates take advantage of this to make various expressions look as much as possible like those on a flat space. The idea is the following. The tangent space  $T_{x_0}M$  looks like a small piece of  $M$  in the neighborhood of  $x_0$ . We can impose linear coordinates on  $T_{x_0}M$ , which is a vector space. Can those coordinates somehow be extended to make coordinates on  $M$  itself?



Let  $V \in T_{x_0}M$  be a vector in the "initial" tangent space (at  $x_0$ ), and consider the geodesic  $x(s)$  that satisfies

$$x(0) = x_0$$

$$\frac{dx}{ds}(0) = V.$$



Let the point reached after elapsed parameter  $s$  be  $p(V, s)$ .

The equation of the geodesic is

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$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\alpha\beta}^\mu(x) \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0.$$

Here  $x^\mu$  = any coordinates  
in a neighborhood of  $x_0$ .

This equation is homogeneous in  $s$ , so if  $x^\mu(s)$  is a solution, then so is  $x^\mu(ks)$  for  $k \in \mathbb{R}$ . But they satisfy different initial conditions,

$$\text{if } \left. \frac{dx^\mu(s)}{ds} \right|_{s=0} = V^\mu, \quad \text{then } \left. \frac{d}{ds} x^\mu(ks) \right|_{s=0} = kV^\mu.$$

In other words, if you scale the initial vector by  $k$ , then the curve is traversed  $k$  times as fast. In other words,

$$p(kV, s/k) = p(V, s).$$

Thus  $p(V, s) = p(sV, 1)$ . The point  $p(V, s)$  actually depends only on the product  $sV$ . Thus we can define,

$$\exp: T_{x_0}M \rightarrow M: V \mapsto p(V, 1).$$

This map is onto in some neighborhood of  $x_0$ , which is "obvious" if you think of  $T_{x_0}M$  being a small piece of  $M$  (for small vectors in  $T_{x_0}M$ ).

Now choose a basis in  $T_{x_0}M$  (could be  $\partial x^\mu|_{x_0}$ ), with respect to which  $V$  has coordinates  $V^\mu$ . Then use  $\exp$  to map these coordinates on  $T_{x_0}M$  onto  $M$  itself. But change the symbol to  $w^\mu$ , to avoid confusion with coords  $V^\mu$  on  $T_{x_0}M$ . Then the geodesics are just radial lines in the  $w^\mu$ -coordinates,

$$w^\mu(t) = t \xi^\mu \quad (\text{eqn. of geodesic in } w^\mu \text{ coords}).$$

The coordinates  $w^\mu$  are called Riemann normal coordinates.

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Various tensor fields simplify in RNC. Consider first  $\Gamma$ . The equ. of a geodesic in RNC is

$$\frac{d^2 W^\mu}{dt^2} + \Gamma_{\alpha\beta}^\mu(t\xi) \frac{dW^\alpha}{dt} \frac{dW^\beta}{dt} = 0$$

$$\text{or } \Gamma_{\alpha\beta}^\mu(t\xi) \xi^\alpha \xi^\beta = 0.$$

Setting  $t=0$  gives  $\Gamma_{\alpha\beta}^\mu(0) \xi^\alpha \xi^\beta = 0.$

This holds for all  $\xi$ , and since  $\Gamma_{\alpha\beta}^\mu$  is symmetric in  $(\alpha\beta)$  (Levi-Civita connection), it follows that

$$\Gamma_{\alpha\beta}^\mu(0) = 0 \quad \text{in RNC.}$$

From this it follows that

$$g_{\mu\nu,\alpha}(0) = 0 \quad \text{since } \nabla g = 0.$$

So if you expand the metric tensor in RNC about  $x_0$ , you find only 2nd order corrections. In general the components  $g_{\mu\nu}$  cannot be constant (that would imply a flat space), but in the right coordinates (namely, RNC) they can be made constant through first order terms in the displacement. The Riemann tensor does not vanish at  $x_0$  (it cannot, in general, since it is a tensor), but the expression in terms of the metric simplifies since  $\Gamma_{\alpha\beta}^\mu(0) = 0$ . You find

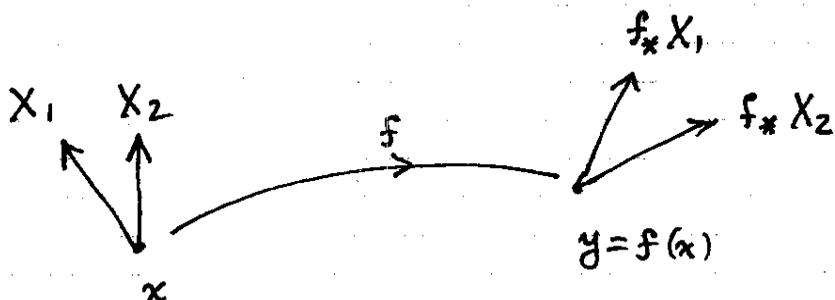
$$R^\mu{}_{\nu\alpha\beta}(0) = \Gamma_{\beta\nu,\alpha}^\mu(0) - \Gamma_{\alpha\nu,\beta}^\mu(0).$$

The vanishing of the Riemann tensor is the integrability condition that there should exist a coordinate system in which  $g_{\mu\nu} = \text{const.}$  This would be a flat space.

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Now for some remarks about conformal transformations and isometries. Let  $(M, g)$  be a (pseudo)-Riemannian manifold, with Levi-Civita connection  $\nabla$ , and consider a map  $f: M \rightarrow M$ . [This discussion is easily generalized to the case  $f: M \rightarrow N$ , between two Riem. manifolds.]. Let  $y = f(x)$ , some  $x \in M$ , and let  $X_1, X_2 \in T_x M$ .



We compare the scalar products  $g|_x(X_1, X_2)$  and  $g|_{f(x)}(f_*X_1, f_*X_2)$ . If these are proportional by some positive scale factor, written  $e^{2\sigma(x)}$  where  $\sigma: M \rightarrow \mathbb{R}$  is a scalar field, i.e., if  $\exists \sigma$  such that

$$g|_{f(x)}(f_*X_1, f_*X_2) = e^{2\sigma(x)} g|_x(X_1, X_2)$$

for all  $X_1, X_2 \in T_x M$ , and all  $x \in M$ , then we say  $f: M \rightarrow M$  is a conformal transformation. The condition can be written more compactly as

$$\boxed{f^*g = e^{2\sigma} g} \quad (\text{Defn of conformal transf. } f)$$

pull back, superscript \*

If this equation holds for  $\sigma = 0$ , i.e., if

$$\boxed{f^*g = g} \quad (\text{Defn. of isometry } f)$$

Then  $f$  is said to be an isometry. An isometry is a special

case of a conformal transformation. Under conformal transformation, scalar products are preserved up to a scaling; this preserves angles but not necessarily lengths. Under an isometry, both lengths and angles are preserved. ⑤ 11/4/08

Historical note: Conformal transformations entered physics with Weyl's 1919 attempt to unify E+M and general relativity (then very new). In Weyl's theory, the integral  $\int A_\mu dx^\mu$  of the E+M vector potential was interpreted as a scale factor for a scaling of the metric. The idea failed, however, because this scale factor is path dependent. But this is where the word "gauge" comes from in "gauge transformation." Later  $\int A_\mu dx^\mu$  was reinterpreted as part of the phase of the quantum wave function, the change of which is still called a gauge transformation.

In modern times conformal field theories are important as exactly solvable 2D models of quantum field theories, in 2D critical phenomena, and in string theory.

An example of a conformal transformation in 2D is any complex function  $w = w(z)$  on the complex plane. Let  $z = x + iy$ ,  $w = u + iv$ . Then can easily show,

$$dx^2 + dy^2 = \frac{du^2 + dv^2}{u_x^2 + u_y^2}, \quad \begin{aligned} u_x &= \frac{\partial u}{\partial x} \\ u_y &= \frac{\partial u}{\partial y} \end{aligned}$$

because of Cauchy-Riemann conditions. Thus the mapping  $z \mapsto w$  is conformal.

Examples of isometries are translations and rotations (the Euclidean group) on Euclidean  $\mathbb{R}^n$ .

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A concept closely related to conformal transformations is the following. Let  $M$  be a manifold with two metrics  $g$  and  $\bar{g}$ , and suppose

$$\bar{g} = e^{2\sigma} g.$$

Then  $g$  and  $\bar{g}$  are said to be conformally related. Let  $\delta$  be a metric that in some coordinates has the form  $\delta_{\mu\nu} = \delta_{\mu\nu}$  (flat space). Then if  $\bar{g} = e^{2\sigma} \delta$ , then  $\bar{g}$  is said to be conformally flat.

As discussed, the integrability condition for the existence of a coordinate system such that  $g_{\mu\nu} = \delta_{\mu\nu}$  is the vanishing of the Riemann tensor,  $R^{\mu\nu\alpha\beta} = 0$ . It turns out that there is another (less restrictive) condition that  $g$  satisfies if it is conformally flat, i.e., if there exists a coord. system and scalar field  $\sigma$  such that  $g_{\mu\nu} = e^{2\sigma} \delta_{\mu\nu}$ . This condition is the vanishing of the Weyl tensor,

$$W_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + \frac{1}{m-2} (g_{\mu\beta} R_{\alpha\nu} - g_{\mu\alpha} R_{\beta\nu} + g_{\alpha\nu} R_{\mu\beta} - g_{\beta\nu} R_{\mu\alpha}) \\ + \frac{1}{(m-1)(m-2)} (g_{\mu\alpha} g_{\beta\nu} - g_{\mu\beta} g_{\alpha\nu}) R,$$

where  $m = \dim M$ ,  $R_{\mu\nu}$  = Ricci tensor,  $R$  = curvature scalar.  $W$  has the property that it is invariant under conformal transformations; hence if  $g$  is conformally flat, then  $W=0$ . Conversely, if  $W=0$ , then for  $m \geq 4$ , the metric is conformally flat; <sup>but</sup> for  $m=3$   $W=0$  always, and for  $m=2$ , any  $g$  is conformally flat.

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Back to isometries. It's easy to show that given  $(M, g)$ , the set of isometries forms a group. This can be thought of as an abstract group  $G$  whose action  $\Phi_a$  on  $M$  is the set of isometries, that is

$$\Phi_a^* g = g, \quad a \in G.$$

( $g =$  metric, not a group element). Assume that  $G$  is a Lie group, and let  $a = \exp(tV)$  where  $V \in \mathfrak{g} =$  Lie algebra of  $G$ . Then

$$\left. \frac{d}{dt} \right|_{t=0} \Phi_{\exp(tV)}^* = V_M = \text{induced vector field} \\ \text{call it } X_V \in \mathfrak{X}(M).$$

This eqn. holds if both sides act on scalars. For other tensors, (such as  $g$ ) replace RHS by  $\mathcal{L}_{X_V}$  (the Lie derivative). Thus if  $G$  is the isometry group, then

$$\Phi_{\exp(tV)}^* g = g,$$

or, applying  $\left. \frac{d}{dt} \right|_{t=0}$ ,

$$\mathcal{L}_{X_V} g = 0.$$

A vector field  $X \in \mathfrak{X}(M)$  such that  $\mathcal{L}_X g = 0$  is called a Killing vector field. Killing vector fields represent infinitesimal isometries. A problem is to find the Killing vector fields given  $g$ .

Let  $X$  be a Killing vector field. Then by writing  $\mathcal{L}_X$  in components, we have

$$(\mathcal{L}_X g)_{\alpha\beta} = X^\mu g_{\alpha\beta,\mu} + X^\mu_{,\alpha} g_{\mu\beta} + X^\mu_{,\beta} g_{\alpha\mu} = 0.$$

This is a differential equation that  $X$  must satisfy. ~~Use the Levi-Civita connection, so that~~ Use the Levi-Civita connection, so that

$$g_{\alpha\beta,\mu} = \Gamma_{\mu\alpha}^\sigma g_{\sigma\beta} + \Gamma_{\mu\beta}^\sigma g_{\alpha\sigma}$$

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Then

~~$$X_{;\alpha}^{\sigma} g_{\sigma\beta} + X_{;\beta}^{\sigma} g_{\alpha\sigma} + \Gamma_{\mu\alpha}^{\sigma} g_{\sigma\beta} X^{\mu} + \Gamma_{\mu\beta}^{\sigma} g_{\mu\alpha} X^{\sigma}$$~~

$$X_{;\alpha}^{\sigma} g_{\sigma\beta} + X_{;\beta}^{\sigma} g_{\alpha\sigma} + \Gamma_{\mu\alpha}^{\sigma} g_{\sigma\beta} X^{\mu} + \Gamma_{\mu\beta}^{\sigma} g_{\mu\alpha} X^{\sigma}$$

$$= g_{\sigma\beta} X_{;\alpha}^{\sigma} + g_{\alpha\sigma} X_{;\beta}^{\sigma}$$

$$= \boxed{X_{\beta;\alpha} + X_{\alpha;\beta} = 0}$$

This is Killing's eqn, nice compact form for eqn. that Killing vectors must satisfy.

Here are some examples of Killing vector fields on some spaces. First take Euclidean  $\mathbb{R}^m$  in standard coordinates. Then  $X_{\beta;\alpha} = X_{\beta\alpha}$ , and Killing's eqn. is

$$X_{\beta;\alpha} + X_{\alpha;\beta} = 0, \text{ also } X_{\beta} = X^{\beta} \text{ since } g_{\mu\nu} = \delta_{\mu\nu}.$$

Expand  $X_{\alpha}$  in a Taylor series:

~~$$X_{\alpha} = a_{\alpha} + b_{\alpha\beta} x^{\beta} + c_{\alpha\beta\gamma} x^{\beta} x^{\gamma} + \dots$$~~

$$X_{\alpha} = a_{\alpha} + b_{\alpha\beta} x^{\beta} + c_{\alpha\beta\gamma} x^{\beta} x^{\gamma} + \dots$$

so

$$X_{\alpha;\beta} = b_{\alpha\beta} + 2 c_{\alpha\beta\gamma} x^{\gamma} + \dots$$

$$X_{\beta\alpha} = b_{\beta\alpha} + 2 c_{\beta\alpha\gamma} x^{\gamma} + \dots$$

$$0 = (b_{\alpha\beta} + b_{\beta\alpha}) + 2(c_{\alpha\beta\gamma} + c_{\beta\alpha\gamma})x^{\gamma} + \dots$$

Thus  $b_{\alpha\beta} = -b_{\beta\alpha}$  (antisymmetric),  $a_{\alpha}$  = any const vector. As for  $c$ , it is symmetric in  $\beta\gamma$  and antisymm. in  $\alpha\beta$ , which  $\Rightarrow c=0$ . Same for all higher tensors. Thus we find

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$$X_\alpha = a_\alpha + b_{\alpha\beta} x^\beta.$$

We ~~may~~ recognize this as an infinitesimal displacement ( $a_\alpha$ ) composed with an infinitesimal rotation ( $b_{\alpha\beta} = -b_{\beta\alpha}$  means  $b \in \mathfrak{so}(n)$ ).

The number of parameters is  $m + \frac{m(m-1)}{2} = \frac{m(m+1)}{2}$  ( $m = \dim M$ ).

This is the number of lin. indep. Killing vector fields; it is a finite number.

In fact, it is the maximum number that a space of  $m$  dimensions can have. For this reason, Euclidean  $\mathbb{R}^m$  is called a maximally symmetric space. Another example of a Max-Sym. Space is the sphere  $S^n$ , with the induced metric obtained by embedding in Euclidean  $\mathbb{R}^{n+1}$ .  $S^n$  is invariant under  $O(n+1)$ , a group with  $n(n+1)/2$  dimensions. (same number). Similarly,  <sup>$n$ -dim.</sup> surfaces of const. neg. curvature can be embedded in Minkowski  $\mathbb{R}^{n+1}$ , and are maximally symmetric under the  $(n+1)$ -dimensional Lorentz group.

Since the Killing vectors are the infinitesimal generators of the group action of the isometry group on  $M$ , they must form a Lie algebra. This is easy to show directly. Let  $X, Y$  be two Killing vector fields, so

$$\mathcal{L}_X g = 0$$

$$\mathcal{L}_Y g = 0.$$

Then since  $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X, Y]}$ , we have  $\mathcal{L}_{[X, Y]} g = 0$ ,

hence  $[X, Y]$  is also a Killing vector field.

Can also talk about conformal Killing vectors.

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New subject. Now we put the basic equations of metrical geometry into a noncoordinate basis, and also introduce the formalism of Cartan. In 4 dimensions, a noncoordinate basis is sometimes called a tetrad or vierbein (German for "four legs"), because a frame is a set of lin. indep. vectors  $\vec{J}$   $\swarrow$  a "dreibein". In many dimensions, we may refer to the frame as a vielbein (many legs).

Nakahara distinguishes components w.r.t. to a vielbein from those w.r.t. a coordinate basis by using  $\alpha, \beta, \gamma, \dots$  for the vielbein and  $\mu, \nu, \lambda, \dots$  for the coordinate basis. We will just use any indices in the following, but it will be understood that we are working in a non-coordinate basis.

Let  $\{e_\mu\}$  be the basis vector fields, assumed to be lin. indep. at each point of some region of space. Let  $\{\theta^\mu\}$  be the dual basis of 1-forms, so that

$$\theta^\mu(e_\nu) = \delta_\nu^\mu.$$

The basis vectors satisfy

$$[e_\mu, e_\nu] = c_{\mu\nu}^\sigma e_\sigma$$

where  $c_{\mu\nu}^\sigma = -c_{\nu\mu}^\sigma$  are the structure constants. (not really const. however).

Similarly, we have

$$d\theta^\mu = -\frac{1}{2} c_{\alpha\beta}^\mu \theta^\alpha \wedge \theta^\beta.$$

(derived previously). (Here's how:

$$\begin{aligned} d\theta^\mu(e_\alpha, e_\beta) &= e_\alpha \left[ \underbrace{\theta^\mu(e_\beta)}_{\delta_\beta^\mu} \right] - e_\beta \left[ \underbrace{\theta^\mu(e_\alpha)}_{\delta_\alpha^\mu} \right] - \theta^\mu([e_\alpha, e_\beta]) \\ &= 0 - 0 - c_{\alpha\beta}^\sigma \theta^\mu(e_\sigma) = -c_{\alpha\beta}^\mu. \end{aligned}$$

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Also use the notation,

$$e_\alpha f = f_{,\alpha} \quad \text{for } f \in \mathcal{F}(M).$$

Note: Nakahara avoids this, he always writes things like  $e_\alpha[\Gamma_{\sigma\tau}^\mu]$  for what I will write as  $\Gamma_{\sigma\tau,\alpha}^\mu$ .

Now begin with  $(M, \nabla)$ , but don't assume any  $g$ , nor that torsion  $T=0$ . ~~Comp~~ First we define connection coefficients,

$$\nabla_\mu \equiv \nabla_{e_\mu}$$

$$\nabla_\mu e_\nu = \Gamma_{\mu\nu}^\alpha e_\alpha$$

Equivalently,  $\nabla_\mu \theta^\nu = -\Gamma_{\mu\alpha}^\nu \theta^\alpha$ .

Now defn of torsion,

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = -T(Y, X) = \text{a vector field.}$$

Components:

$$\begin{aligned} T(e_\mu, e_\nu) &= T_{\mu\nu}^\alpha e_\alpha \\ &= \nabla_\mu e_\nu - \nabla_\nu e_\mu - [e_\mu, e_\nu] \\ &= (\Gamma_{\mu\nu}^\alpha - \Gamma_{\nu\mu}^\alpha - C_{\mu\nu}^\alpha) e_\alpha \end{aligned}$$

hence

$$T_{\mu\nu}^\alpha = \Gamma_{\mu\nu}^\alpha - \Gamma_{\nu\mu}^\alpha - C_{\mu\nu}^\alpha$$

Similarly for the curvature tensor,

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$$

~~$$R(e_\alpha, e_\beta) e_\nu = R^\mu{}_{\nu\alpha\beta} e_\mu$$~~

$$R(e_\alpha, e_\beta) e_\nu = R^\mu{}_{\nu\alpha\beta} e_\mu$$

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We found previously,

$$R^{\mu}_{\nu\alpha\beta} = \Gamma^{\mu}_{\beta\nu,\alpha} - \Gamma^{\mu}_{\alpha\nu,\beta} + \Gamma^{\sigma}_{\beta\nu} \Gamma^{\mu}_{\alpha\sigma} - \Gamma^{\sigma}_{\alpha\nu} \Gamma^{\mu}_{\beta\sigma} - C^{\sigma}_{\alpha\beta} \Gamma^{\mu}_{\sigma\nu}.$$

Now follow Cartan and make the following definitions:

$$\omega^{\mu}_{\nu} = \Gamma^{\mu}_{\alpha\nu} \theta^{\alpha} \quad (\text{Lie-algebra valued 1-form})$$

$$T^{\mu} = \frac{1}{2} T^{\mu}_{\alpha\beta} \theta^{\alpha} \wedge \theta^{\beta} \quad (\text{vector-valued 2-form})$$

$$R^{\mu}_{\nu} = \frac{1}{2} R^{\mu}_{\nu\alpha\beta} \theta^{\alpha} \wedge \theta^{\beta} \quad (\text{Lie-algebra valued 2-form}).$$

Now take

$$d\theta^{\mu} = -\frac{1}{2} C^{\mu}_{\alpha\beta} \theta^{\alpha} \wedge \theta^{\beta}$$

use components of  $T$ , elim.  
 $C^{\mu}_{\alpha\beta}$  in favor of  $T, \Gamma$ ,

$$= +\frac{1}{2} \left( T^{\mu}_{\alpha\beta} - \Gamma^{\mu}_{\alpha\beta} + \Gamma^{\mu}_{\beta\alpha} \right) \theta^{\alpha} \wedge \theta^{\beta}$$

$$= T^{\mu} - \omega^{\mu}_{\beta} \wedge \theta^{\beta},$$

or

$$\boxed{d\theta^{\mu} + \omega^{\mu}_{\beta} \wedge \theta^{\beta} = T^{\mu}}$$

1st Cartan structure eqn.

LHS is a kind of covariant derivative of a 1-form.

Next, take defn.  $\omega^{\mu}_{\nu} = \Gamma^{\mu}_{\alpha\nu} \theta^{\alpha}$ , apply  $d$ :

$$d\omega^{\mu}_{\nu} = d(\Gamma^{\mu}_{\alpha\nu} \theta^{\alpha}) = \Gamma^{\mu}_{\alpha\nu,\beta} \theta^{\beta} \wedge \theta^{\alpha} + \Gamma^{\mu}_{\alpha\nu} d\theta^{\alpha}$$

$$= \frac{1}{2} \left( \Gamma^{\mu}_{\alpha\nu,\beta} - \Gamma^{\mu}_{\beta\nu,\alpha} \right) \theta^{\beta} \wedge \theta^{\alpha} - \frac{1}{2} \Gamma^{\mu}_{\alpha\nu} C^{\alpha}_{\sigma\tau} \theta^{\sigma} \wedge \theta^{\tau}$$