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10/30/08

$$\begin{aligned} T_1 &= \nabla_x (Yf)Z + \nabla_x f \nabla_Y Z \\ &= (XYf)Z + (Yf)\nabla_x Z + (Xf)\nabla_Y Z + f \nabla_x \nabla_Y Z \end{aligned}$$

$T_2 = T_1$  with  $X \leftrightarrow Y$ , minus sign.

$$= -(YXf)Z - (Xf)\nabla_Y Z - (Yf)\nabla_X Z - f \nabla_Y \nabla_X Z$$

$$T_3 = -([X, Y]f)Z - f \nabla_{[X, Y]} Z$$

add em up, get  $f R(X, Y)Z$ .

So  $R$  is a tensor.

Now compute the components of  $R$  (starting from coordinate-free definition) and compare to earlier coordinate-based calculation.

For variety do this in a non-coordinate basis  $e_\mu$  ( $\neq \frac{\partial}{\partial x^\mu}$ ).

Define:

①  $f_{, \mu} = (e_\mu f)$  when  $f \in \mathcal{F}(M)$ , generalizes notation  $f_{, \mu} = e_\mu f = \frac{\partial f}{\partial x^\mu}$  for a coordinate basis.

②  $\nabla_\mu = \nabla_{e_\mu}$

③  $\nabla_\alpha e_\beta = \Gamma_{\alpha\beta}^\mu e_\mu$

④  $R(e_\alpha, e_\beta)e_\nu = R^\mu{}_{\nu\alpha\beta} e_\mu$

(2)

10/30/08

$$R(e_\alpha, e_\beta) e_\nu = \underbrace{\nabla_\alpha \nabla_\beta e_\nu} - \underbrace{\nabla_\beta \nabla_\alpha e_\nu} - \underbrace{\nabla [e_\alpha, e_\beta] e_\nu}$$

$$\rightarrow = \nabla_\alpha (\Gamma_{\beta\nu}^\sigma e_\sigma) = \Gamma_{\beta\nu, \alpha}^\sigma e_\sigma + \Gamma_{\beta\nu}^\sigma \Gamma_{\alpha\sigma}^\mu e_\mu$$

$$\rightarrow = \text{same w. } (\alpha \leftrightarrow \beta).$$

$$\rightarrow = - \nabla_{C_{\alpha\beta}^\sigma e_\sigma} e_\nu = - C_{\alpha\beta}^\sigma \nabla_\sigma e_\nu = - C_{\alpha\beta}^\sigma \Gamma_{\sigma\nu}^\mu e_\mu$$

$C_{\alpha\beta}^\sigma =$  structure consts of basis.

gives

$$R^\mu{}_{\nu\alpha\beta} = \Gamma_{\beta\nu, \alpha}^\mu + \Gamma_{\beta\nu}^\sigma \Gamma_{\alpha\sigma}^\mu - \Gamma_{\alpha\nu, \beta}^\mu - \Gamma_{\alpha\nu}^\sigma \Gamma_{\beta\sigma}^\mu - C_{\alpha\beta}^\sigma \Gamma_{\sigma\nu}^\mu$$

agrees with earlier calculation in coord. basis, for which  $C_{\alpha\beta}^\sigma = 0$ .

In the case of the Levi-Civita connection on a (pseudo)-Riemannian manifold, the curvature tensor is <sup>also</sup> called the Riemann tensor. I'm not sure if that is appropriate in other cases.

The curvature tensor has various symmetries, depending on the assumptions. In the most general case (manifold  $M$  + connection  $\nabla$ , nothing else) we have the symmetry,

$$R(X, Y) = -R(Y, X) \quad \text{or} \quad R^\mu{}_{\nu\alpha\beta} = -R^\mu{}_{\nu\beta\alpha}.$$

This just says that  $R$  is a 2-form (indices  $\alpha, \beta$ ). That's all in this case.

③

10/30/08

If in addition we assume torsion  $T=0$ , then there are two further symmetries, called the 1st and 2nd Bianchi identities. The first Bianchi identity is an algebraic condition on  $R$ . In components it is

$$R^{\mu}{}_{\nu\alpha\beta} = 0 \quad \text{where } [ ] \text{ means complete antisymmetrization,}$$

i.e.  $R^{\mu}{}_{\nu\alpha\beta} + R^{\mu}{}_{\alpha\beta\nu} + R^{\mu}{}_{\beta\nu\alpha} = 0$  in this case, since  $R$  is already antisymmetric in  $(\alpha\beta)$ . This statement is equivalent to

$$R(X, Y)Z + \text{cyclic} = 0$$

where  $\text{cyclic}$  means to cycle  $X, Y, Z$  ( $X, Y$  correspond to indices  $\alpha\beta$ , the 2-form indices, while  $Z$  corresponds to  $\nu$ , in  $R^{\mu}{}_{\nu\alpha\beta}$ ). We will prove this in coordinate free form. First use the definition of  $R(X, Y)$  to write

$$R(X, Y)Z + \text{cyclic} = \nabla_X \boxed{\nabla_Y Z} - \nabla_Y \nabla_X Z + \text{cyclic}.$$

↑ substitute

But since  $T=0$ ,  $T(Y, Z) = \nabla_Y Z - \nabla_Z Y - [Y, Z] = 0$ , so

$$\text{RHS} = \nabla_X \nabla_Y Z + \nabla_X [Y, Z] - \nabla_Y \nabla_X Z - \nabla [X, Y] Z + \text{cyclic}$$

$\begin{array}{c} \text{X} \rightarrow \text{Y} \rightarrow \text{Z} \\ \uparrow \\ \text{becomes } \nabla_Y \nabla_X Z \end{array}$ 
↙ cancel.

$$\text{RHS} = \nabla_X [Y, Z] - \nabla [X, Y] Z + \text{cyclic}$$

But  $T(X, [Y, Z]) = \nabla_X [Y, Z] - \nabla_{[Y, Z]} X - [X, [Y, Z]] = 0$  so

$$\text{RHS} = \nabla_{[Y, Z]} X - \nabla [X, Y] Z + [X, [Y, Z]] + \text{cyclic}$$

$\begin{array}{c} \text{Jacobi} \\ \swarrow \quad \searrow \\ \text{cancel} \quad \hookrightarrow \nabla_{[Y, Z]} X \end{array}$

= 0 QED.

④

10/30/08

The 2nd Bianchi identity is a differential equation satisfied by  $R^\mu{}_\nu\alpha\beta$ . Since  $R$  is a 2-form (indices  $\alpha\beta$ ), we might ask what the exterior derivative is and whether it is interesting. If  $R$  were a scalar-valued 2-form instead of a Lie-algebra-valued 2-form, we might compute (in components)

$$R^\mu{}_\nu[\alpha, \beta, \gamma]$$

which would be the components of something like " $\mathbb{L}^4$ ". But since  $R$  is a Lie-algebra-valued form, it turns out we must replace the comma by a semicolon.

Digression on semicolon notation. If  $T$  is a type  $(r, s)$  tensor and  $X$  is a vector field, then  $\nabla_X T$  is also a type  $(r, s)$  tensor. But the resulting object is point-wise linear in  $X$ , so ~~if~~ removing the  $X$  it corresponds to a type  $(r, s+1)$  tensor, call it  $\nabla T$ , such that

$$\nabla_X T = i_X(\nabla T)$$

where  $i_X$  means, "contract  $X$  with one index of  $\nabla T$ ". For example, if  $T=Y$  = a vector field = a type  $(1, 0)$  tensor, then  $\nabla Y$  is a type  $(1, 1)$  tensor, and we write its components by

$$\begin{aligned} \nabla Y^\mu{}_\alpha &= (\nabla_\alpha Y)^\mu = \theta^\mu(\nabla_{e_\alpha} Y) \\ &\equiv Y^\mu{}_{;\alpha} \end{aligned}$$

Thus

$$\nabla_X Y = X^\alpha Y^\mu{}_{;\alpha} \quad \text{and}$$

$$Y^\mu{}_{;\alpha} = Y^\mu{}_{,\alpha} + \Gamma^\mu_{\alpha\beta} Y^\beta.$$

⑤

10/30/08

To return to the 2nd Bianchi identity, it is

$$R^{\mu}{}_{\nu[\alpha\beta;\gamma]} = 0.$$

In coordinate free form this is

$$(\nabla_Z R)(X, Y) + \text{cyclic} = 0$$

where  $X, Y$  correspond to  $\alpha\beta$  and  $Z$  to  $\gamma$ . To prove this, let  $W$  be a vector field so that  $R(X, Y)W$  is another vector field, apply  $\nabla_Z$  to the latter and use Leibnitz:

swap, change sign  
↕

$$\nabla_Z R(X, Y)W = (\nabla_Z R)(X, Y)W + R(\nabla_Z X, Y)W + R(X, \nabla_Z Y)W + R(X, Y)\nabla_Z W.$$

Now cancel  $W$  to get an equation among operators acting on vector fields, and rewrite it as

$$(\nabla_Z R)(X, Y) = [\nabla_Z, R(X, Y)] - R(\nabla_Z X, Y) + R(\nabla_Z Y, X)$$

Now add cyclic perms and use defn of  $R(X, Y)$ ,

$$(\nabla_Z R)(X, Y) + \text{cyclic} = [\nabla_Z, [X, Y]] - [R(\nabla_Z X, Y) - R(\nabla_Z Y, X) + \text{cyclic}]$$

o Jacobi

Now use  $T(Z, X) = 0 \Rightarrow \nabla_Z X = \nabla_X Z + [Z, X]$

$$\begin{aligned} \text{RHS} &= -[\nabla_Z, \nabla_{[X, Y]}] - R(\nabla_X Z, Y) - R([Z, X], Y) + R(\nabla_Z Y, X) \\ &= -[\nabla_Z, \nabla_{[X, Y]}] - R(\nabla_{[X, Z]}, Y) - \nabla_{[X, Z]} Y + \text{cyclic} \\ &= -[\nabla_Z, \nabla_{[X, Y]}] - R(\nabla_{[X, Z]}, Y) + \nabla_{[X, Z]} Y + \text{cyclic} = 0 \text{ QED} \end{aligned}$$

cancel use defn of R o Jacobi

⑥

10/30/08

Now we turn to Riemannian manifolds. These are manifolds that possess a metric tensor. We have already discussed metric tensors on a vector space  $V$ ; now we promote this idea into a field, by identifying the former  $V$  with  $T_x M$  (one metric in each tangent space). The result is a type  $(0,2)$  tensor  $g$ ,

$$g|_x: T_x M \times T_x M \rightarrow \mathbb{R} \quad (\text{at a point})$$

$$g: \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{F}(M) \quad (\text{as a field}).$$

At each point  $x \in M$ ,  $g$  satisfies the requirements of a metric:

$$(1) \quad g(X, Y) = g(Y, X) \quad (\text{symmetric})$$

$$(2) \quad g \text{ is nonsingular.}$$

Property (2) can be expressed in terms of the component matrix  $g_{\mu\nu}$  of  $g$  w.r.t. some basis  $\{e_\mu\}$ ,

$$(2') \quad \det(g_{\mu\nu}) \neq 0,$$

a stmt independent of basis. By an orthogonal change of basis,  $g_{\mu\nu}$  can be diagonalized. The eigenvalues are not invariant under change of basis (which generally need not be orthogonal), but their signs are. In fact, by scaling the basis vectors by a factor  $a \neq 0$ , after  $g_{\mu\nu}$  has been diagonalized, the eigenvalues are scaled by  $a^2 > 0$ . Thus they can be scaled to  $\pm 1$  ( $0$  is excluded since  $g_{\mu\nu}$  is nonsingular). The number of  $+1$ 's and  $-1$ 's in this final form is an invariant property of  $g_{\mu\nu}$ . ~~Of all~~ The list of these numbers is the signature of  $g$ .

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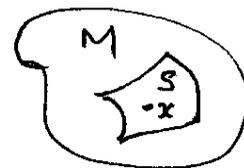
10/30/08

If the signature contains only +1's, then  $g_{\mu\nu}$  is positive definite, and  $M$  is said to be a Riemannian manifold. If some are +1 and others -1, then  $M$  is pseudo-Riemannian. The signature of  $g$  cannot change as we move around on  $M$  because eigenvalues are not allowed to pass through 0.

Let  $S$  be a submanifold of  $M$ , a (pseudo)-Riemannian manifold. Then  $g$  (on  $M$ ) can be restricted to  $S$ , creating a type  $(0,2)$  tensor on  $S$ . (In fact, any purely covariant tensor, a 2-form for example, can be restricted to submanifolds in the same manner.)

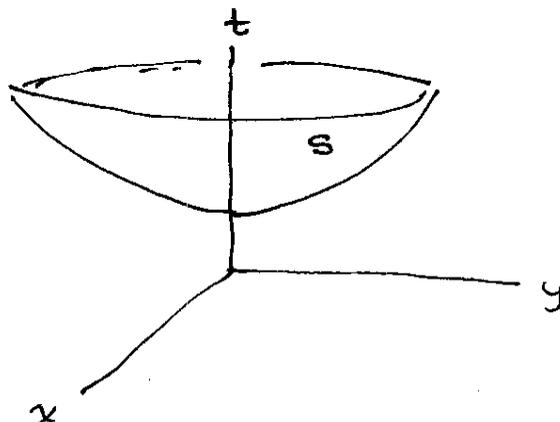
We just define

$$g|_S(x, Y) = g(x, Y),$$



where  $X, Y \in T_x S$ , are reinterpreted as elements of  $T_x M$ .  $g|_S$  is then a tensor field on  $S$ . If  $g$  is positive definite (Riemannian case), then  $g|_S$  is also, and  $S$  becomes a Riemannian manifold (every submanifold is Riemannian). But if  $M$  is pseudo-Riemannian, then  $g|_S$  ~~is~~ is not necessarily nonsingular everywhere.

Example: Let  $M = \mathbb{R}^4$  with Minkowski metric, let  $S =$  unit hyperbola (or mass shell):



③

10/30/08

Metric on  $M$ :

$$-dt^2 + dx^2 + dy^2 + dz^2$$

Metric on  $S$ : turns out to be Riemannian (pos. def.) but not flat,  
 $S$  is surface of const. negative curvature (Lobachevskian plane).

Let  $x^{\mu}$  be coordinates. Then in the coordinate basis,

$g$  is

$$g = g_{\mu\nu} dx^{\mu} \otimes dx^{\nu}, \text{ often written without the } \otimes.$$

for short.

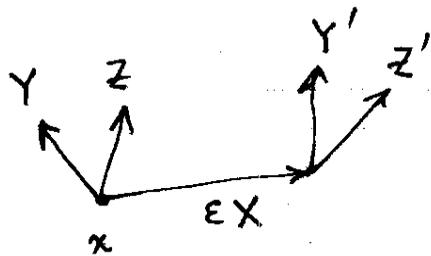
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Now suppose the manifold has both a connection and a metric. We emphasize that a metric and a connection are two different geometrical constructions. You can have a manifold with a connection but without a metric. However if you do have both, you can compute  $\nabla g$ .

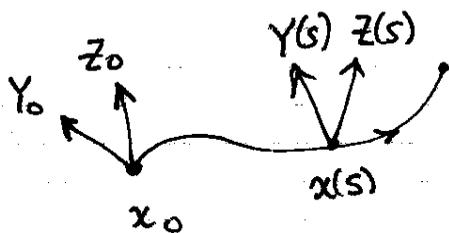
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10/30/08

A ~~metric~~ connection for which  $\nabla_X g = 0$  (for all  $X$ ) is called a metric connection. If you have a metric connection, then the scalar product of parallel transported vectors is ~~fixed~~ constant: Let  $Y, Z \in T_x M$  be two vectors parallel transported along  $\varepsilon X$  to a new point  $x + \varepsilon X$ , to give new vectors  ~~$Y, Z$~~   $Y', Z'$ :



Then  $g(Y, Z) = g(Y', Z')$ , if you use a metric connection. Similarly, if  $Y(s), Z(s)$  are the parallel transports of  $Y_0, Z_0 \in T_{x_0} M$  along a curve,



then

$$\frac{d}{ds} [Y(s)_\mu Z(s)^\mu] = 0.$$

The condition  $\nabla_X g = 0$ ,  $\forall X$ , implies:

$$g_{\mu\nu,\alpha} = \Gamma_{\alpha\mu}^\beta g_{\beta\nu} + \Gamma_{\alpha\nu}^\beta g_{\mu\beta}.$$

This equ. can be solved ~~in terms~~ for  $\Gamma$  in terms of  $g$  and its derivatives and the torsion tensor. First define

$$\Gamma_{\mu\alpha\beta} = g_{\mu\nu} \Gamma_{\alpha\beta}^\nu.$$

10/30/08

Then write

$$S_{\mu\alpha\beta} = \Gamma_{\mu\alpha\beta} + \Gamma_{\mu\beta\alpha}$$

$$T_{\mu\alpha\beta} = \Gamma_{\mu\alpha\beta} - \Gamma_{\mu\beta\alpha}$$

these are the symmetric and antisymmetric parts of  $\Gamma$ . The antisymmetric part is the same as the torsion tensor (but the symmetric part is not a tensor). So we have

$$\Gamma_{\mu\alpha\beta} = \frac{1}{2}(S_{\mu\alpha\beta} + T_{\mu\alpha\beta})$$

$$g_{\mu\nu,\alpha} = \Gamma_{\nu\alpha\mu} + \Gamma_{\mu\alpha\nu}$$

$$= \frac{1}{2}(S_{\nu\alpha\mu} + S_{\mu\alpha\nu} + T_{\nu\alpha\mu} + T_{\mu\alpha\nu})$$

$$g_{\nu\alpha,\mu} = \frac{1}{2}(S_{\alpha\mu\nu} + S_{\nu\mu\alpha} + T_{\alpha\mu\nu} + T_{\nu\mu\alpha})$$

$$g_{\alpha\mu,\nu} = \frac{1}{2}(S_{\mu\nu\alpha} + S_{\alpha\nu\mu} + T_{\mu\nu\alpha} + T_{\alpha\nu\mu})$$

Solve for  $S_{\alpha\mu\nu} = S_{\alpha\nu\mu}$ 

$$g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha} = \underbrace{S_{\alpha\mu\nu}}_{\rightarrow 2\Gamma_{\alpha\mu\nu} - T_{\alpha\mu\nu}} + T_{\mu\nu\alpha} + T_{\nu\mu\alpha},$$

so, 
$$\Gamma^{\beta}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}) + \frac{1}{2} (T^{\beta}_{\mu\nu} + T_{\nu}^{\beta\mu} + T_{\nu}^{\beta\mu})$$

$\rightarrow$  denoted  $\{\Gamma^{\beta}_{\mu\nu}\} =$  Christoffel symbols.

10/30/08

As claimed, we have  $\Gamma$  in terms of  $g$  and the torsion, for a metric connection. If the torsion vanishes, then

$$\Gamma_{\alpha\beta}^{\mu} = \{\alpha\beta\}^{\mu} = \Gamma_{\beta\alpha}^{\mu}.$$

The connection that satisfies this is a special metric connection, called the Levi-Civita connection. In a sense it is the simplest metric connection.

Under the Levi-Civita connection, a connection geodesic is the same as a metric geodesic, i.e.,

$$0 = \delta \int ds = \delta \int \sqrt{g_{\mu\nu} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds}} ds$$

$$\Rightarrow \frac{d^2 x^{\mu}}{ds^2} + \{\alpha\beta\}^{\mu} \frac{dx^{\alpha}}{ds} \frac{dx^{\beta}}{ds} = 0.$$

Under a metric connection, the parallel transport of vectors around a loop preserves scalar products, so the holonomy is an orthogonal transformation  $O(n)$  for a Riemannian manifold, or an element of  $O(r,s)$  for a pseudo-Riemannian manifold. For example, in GR, the holonomy is an element of the Lorentz group  $O(3,1)$  of special relativity.

Under a metric connection, the curvature tensor has further symmetries. With  $g$  we can raise and lower indices, and talk about  $R_{\mu\nu\alpha\beta}$ . The  $\mu\nu$  indices now refer to an element of the Lie algebra of an orthogonal group, so they are anti-symmetric, i.e.

$$R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta}.$$

This only requires  $\nabla g = 0$ , not  $T = 0$ . But...

(12)

10/30/08

Finally, if we assume both a metric connection  $\nabla g = 0$  and vanishing torsion  $T = 0$ ,  $\rightarrow$  i.e., the Levi-Civita connection, then there is another symmetry,

$$R_{\mu\nu\alpha\beta} = R_{\beta\mu\nu\alpha}.$$

You can compute the number of independent components of  $R^{\mu\nu\alpha\beta}$  under various assumption. For example, if you just have a connection and nothing else there are

$$\frac{m^3(m-1)}{2} \quad m = \dim M.$$

indep components (only symmetry is 2-form symmetry). But with the Levi-Civita connection (do the combinatorics) you find

$$\frac{m^2(m^2-1)}{12}$$

indep components. Table:

$m$	$\frac{m^2(m^2-1)}{12}$
0	0
1	0
2	1
3	6
4	20

Easy to understand why only one component for  $m=2$ . A 2-form on a 2D space has only one indep. components, and  $SO(2)$  is 1-dimensional. In this case, the holonomy around any loop (infinitesimal or otherwise) is specified by an angle of rotation  $\theta$ , and  $R$  can be reduced to a scalar-valued 2-form (the usual kind). Thus, it represents a kind of "angle density" on  $M$ , the integral of this over some area gives the holonomy

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10/30/08

on parallel transporting around the boundary. For example, on the usual 2-sphere with the obvious metric and connection, you find

$$R = \sin\theta d\theta \wedge d\phi = "d\Omega"$$

the solid angle.

In higher dimensions the holonomy group is usually non Abelian, so you can't get the ~~angle of~~ holonomy on going around a finite loop by integrating the 2-form (R) over the interior.

→ Assume LC connection.

Some tensors important in GR. By contracting the Riemann tensor, we get the Ricci tensor, which can be contracted to the curvature scalar:

Ricci:  $R_{\nu\beta} = R^{\mu}{}_{\nu\mu\beta}$

Curv. Scalar:  $R = R^{\nu}{}_{\nu} = R^{\mu\nu}{}_{\mu\nu}$ .

The 2nd Bianchi identity implies a differential equation satisfied by these:

$$R^{\mu}{}_{\nu\alpha\beta;\gamma} + R^{\mu}{}_{\nu\gamma\alpha;\beta} + R^{\mu}{}_{\nu\beta\gamma;\alpha} = 0 \quad (\text{contract } \mu, \alpha)$$

$$R_{\nu\beta;\gamma} + \underbrace{R^{\mu}{}_{\nu\gamma\mu;\beta}} + R^{\mu}{}_{\nu\beta\gamma;\mu} = 0$$

$$\hookrightarrow = -R^{\mu}{}_{\nu\mu\gamma;\beta}$$

$$= -R_{\nu\gamma;\beta}$$

$$R_{\nu\beta;\gamma} + R_{\nu\gamma;\beta} + R^{\mu}{}_{\nu\beta\gamma;\mu} = 0. \quad (\text{contract } \nu, \beta)$$

$$R_{;\gamma} - R^{\nu}{}_{\gamma;\nu} + \underbrace{R^{\mu\nu}{}_{\nu\gamma;\mu}} = 0$$

$$\hookrightarrow = -R^{\mu\nu}{}_{\nu\gamma;\mu} = -R^{\mu}{}_{\gamma;\mu}$$

$$\Rightarrow R^{\mu}{}_{\nu;\mu} - \frac{1}{2} R_{;\nu} = 0.$$