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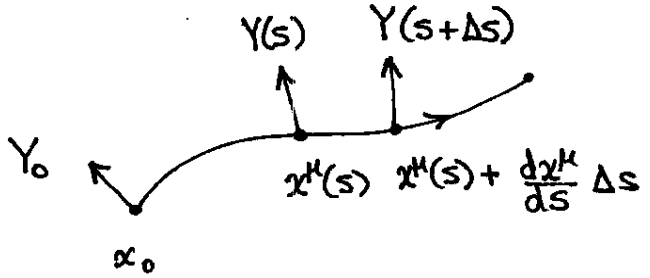

We let $Y(s)$ be the parallel transported vector, $Y(s) \in T_{x(s)}M$.
Then

$$Y^\mu(s+\Delta s) = Y^\mu(s) + \frac{dY^\mu}{ds}\Delta s$$

$$= Y^\mu(s) - \Gamma^\mu_{\alpha\beta}\left(\frac{dx^\alpha}{ds}\Delta s\right) Y^\beta.$$

$$\Rightarrow \quad \boxed{\frac{dY^\mu}{ds} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}Y^\beta = 0}$$

Eqn. of parallel transport.
homog. in s, transport indep. of param. s of curve.

a differential eqn. that can be solved subject to init conds. $Y(0) = Y_0$.
Can also write it,

$$\boxed{\nabla_{\frac{d}{ds}} Y = 0}$$

where $\frac{d}{ds}$ is the tangent vector along the curve.

An interesting vector to parallel transport is the tangent vector itself. If the parallel transport of the tangent vector is the same as the tangent vector itself, then this is a special property of the curve. Then we have

$$\boxed{\begin{aligned} &\nabla_{\frac{d}{ds}}\frac{d}{ds} = 0 \\ \text{or} \quad &\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \end{aligned}}$$

Eqn. of a <u>geodesic</u>.

2nd order ode, requires $x^\mu(0)$, $\frac{dx^\mu}{ds}(0)$.

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More exactly, this is what may be called a "connection geodesic." If the space is a Riemannian manifold, you can also have a metrical geodesic, the shortest curve between two points. These two need not be the same (indeed M can have a connection without having a metric). But for special connections on Riemannian manifolds, the two kinds of geodesics are identical.

Now we extend ∇ to other types of tensors (besides vectors). We postulate:

$$\nabla_X (T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2) \qquad \text{(Leibnitz)}$$

and $\quad \nabla_X \delta = 0 \qquad$ (Kronecker δ).

For example, with $T_1 = f$ (a scalar) and $T_2 = Y$ (a vector field), we have

$$\nabla_X (fY) = (\nabla_X f) Y + f \nabla_X Y,$$

which, comparing to results above, shows that

$$\boxed{\nabla_X f = Xf}$$

(for scalars, the covariant derivative is the obvious convective derivative).

Next, we can work out the action of ∇_X on a 1-form ω by using the rules above:

$$\nabla_X [\omega(Y)] = (\nabla_X \omega)(Y) + \omega(\nabla_X Y)$$

$$\text{LHS} = X[\omega(Y)] = X^\mu (\omega_\nu Y^\nu)_{,\mu} = X^\mu (\omega_{\nu,\mu} Y^\nu + \omega_\nu Y^\nu_{,\mu})$$

cancel

$$\text{RHS} = (\nabla_X \omega)_\nu Y^\nu + \omega_\nu X^\mu (Y^\nu_{,\mu} + \Gamma^\nu_{\mu\alpha} Y^\alpha).$$

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So we can solve for $(\nabla_X \omega)_\nu$, get

$$\boxed{(\nabla_X \omega)_\nu = X^\mu (\omega_{\nu,\mu} - \Gamma^\alpha_{\mu\nu}\, \omega_\alpha)}$$

c.f. earlier result for vectors

$$\boxed{(\nabla_X Y)^\nu = X^\mu (Y^\nu_{,\mu} + \Gamma^\nu_{\mu\alpha}\, Y^\alpha)}$$

Similarly can work out rules for covariant derivatives (in components) for an arbitrary tensor. Basically you get an ordinary derivative with one correction term with Γ and a + sign for every contravariant index, and one correc. term with Γ and a − sign for every covariant index. For example, you find for the metric tensor,

$$(\nabla_X g)_{\mu\nu} = X^\alpha (g_{\mu\nu,\alpha} - \Gamma^\beta_{\alpha\mu}\, g_{\beta\nu} - \Gamma^\beta_{\alpha\nu}\, g_{\mu\beta})$$

Note, also have

$$\boxed{\nabla_\mu\, dx^\nu = -\Gamma^\nu_{\mu\alpha}\, dx^\alpha}$$

Now we turn to the <u>transformation properties</u> of the connection coefficients $\Gamma^\mu_{\alpha\beta}$. Basic fact is that $\Gamma^\mu_{\alpha\beta}$ is <u>not a tensor</u>. A tensor is a mapping of vectors and covectors onto scalars, that is point-wise linear (linear at each point). We can think of Γ as such a mapping,

$$\Gamma : \mathfrak{X}^*(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{F}(M) : (\alpha, X, Y) \mapsto \alpha(\nabla_X Y),$$

$$\Gamma^\mu_{\alpha\beta} = dx^\mu (\nabla_\alpha e_\beta) \qquad e_\beta = \frac{\partial}{\partial x^\beta}.$$

But it is not point-wise linear in the Y operand (it depends on

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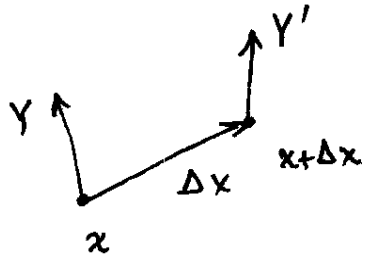
the derivatives of Y as well as the value of Y at a point). Here are various ways to see this.

① Consider the parallel transport of Y from x to $x+\Delta x$,



We have

$$Y'^{\mu} = \underbrace{\left(\delta^{\mu}_{\nu} + \Delta x^{\alpha}\, \Gamma^{\mu}_{\alpha\nu}\right)}\, Y^{\nu}$$

$\hookrightarrow$ a near-identity element of $GL(n,\mathbb{R})$, so we can think of

$\Gamma^{\mu}_{\cdot\,\nu} = dx^{\alpha}\, \Gamma^{\mu}_{\cdot\,\nu}$ as a <u>$\mathfrak{gl}(n,\mathbb{R})$-valued 1-form</u>.

But notice that the <u>components</u> of this 1-form ~~[illegible]~~ depend on the basis chosen in <u>two different</u> tangent spaces (at x and $x+\Delta x$). You can change one without changing the other. Hence $\Gamma^{\mu}_{\alpha\nu}$ does not transform as a tensor.

To emphasize this, consider the following fact: The <u>difference</u> between two connections, say, $\Gamma - \bar{\Gamma}$, <u>is a tensor</u>. That's because $\Gamma - \bar{\Gamma}$ can be thought of as specifying the parallel transport from x to $x+\Delta x$, using Γ, then back again, using $\bar{\Gamma}$. The vector is transported from one tangent space back to the <u>same</u> tangent space. (Say, $Y \to Y' \to Y''$). Then

$$Y''^{\mu} = \left(\delta^{\mu}_{\nu} + dx^{\alpha}\, \Gamma^{\mu}_{\alpha\nu} - dx^{\alpha}\, \bar{\Gamma}^{\mu}_{\alpha\nu}\right) Y^{\nu}.$$

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Thus, only one basis (in T_xM) need be chosen to specify the near-identity element of $GL(n,\mathbb{R})$ mapping Y to Y''.

② Just do a brute-force transformation of the connection coefficients. Let

$$e_\alpha = \frac{\partial}{\partial x^\alpha} \qquad e'_\mu = \frac{\partial}{\partial x'^\mu}, \qquad \nabla_\alpha = \nabla_{e_\alpha} \qquad \nabla'_\mu = \nabla_{e'_\mu}$$

$$\Gamma^\alpha_{\beta\gamma} = (dx^\alpha, \nabla_\beta e_\gamma)$$

$$= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial^2 x'^\mu}{\partial x^\beta \partial x^\gamma} + \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x'^\sigma}{\partial x^\gamma}\frac{\partial x'^\nu}{\partial x^\beta}\Gamma'^\mu_{\nu\sigma}$$

The 2nd term looks like a tensor transformation law, but the first term spoils it (and involves 2nd derivatives of the coordinate transformation). But if you subtract the transformation laws for two Γ's, say, $\Gamma - \bar{\Gamma}$, then the first term cancels.

Transformation laws like this are familiar for the gauge potential A^a_μ of gauge-field theories (Yang-Mills, QCD).

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besides subtracting $\Gamma - \bar{\Gamma}$

Notice that another way to cancel the first term is to <u>antisymmetrize</u> in (β,γ). This leads to a tensor called the <u>torsion</u>:

$$T^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma} - \Gamma^\alpha{}_{\gamma\beta}.$$

This is the <u>component</u> definition of the torsion. The <u>coordinate-free</u> definition is

$$T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) : (X, Y) \mapsto \nabla_X Y - \nabla_Y X - [X, Y]$$ 10/28/08

This is obviously antisymmetric. To show that it is a tensor, we must show that it is pointwise linear in each operand (but due to the antisymmetry, we need only check one). Let $f \in \mathcal{F}(M)$. Then

$$T(fX, Y) = \nabla_{fX} Y - \nabla_Y (fX) - [fX, Y]$$

$$= f \nabla_X Y - (Yf)X - f \nabla_Y X - f XY + (Yf)X + f YX$$

cancel

$$= f \left[\nabla_X Y - \nabla_Y X - [X, Y] \right]$$

$$= f\, T(X, Y).$$

So it's a tensor. Now let $\{e_\mu\}$ be any basis of vector fields (coordinate or non-coordinate). Then we define the components of T by

$$T(e_\alpha, e_\beta) = T^\mu{}_{\alpha\beta}\, e_\mu$$

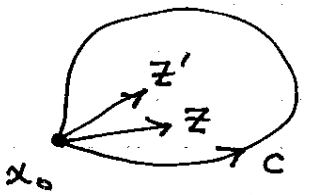
$$= \nabla_\alpha e_\beta - \nabla_\beta e_\alpha - [e_\alpha, e_\beta]$$

$$= \left(\Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha} - c^\mu_{\alpha\beta} \right) e_\mu,$$

where $c^\mu_{\alpha\beta}$ are the "structure constants", i.e. $[e_\alpha, e_\beta] = c^\mu_{\alpha\beta}\, e_\mu$.

(Really $c^\mu_{\alpha\beta}$ depend on x, in general). This agrees with earlier coordinate-based definition of T, where we used a coordinate basis so that $c^\mu_{\alpha\beta} = 0$.

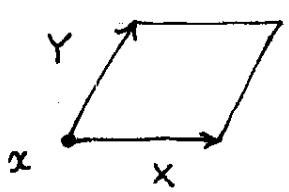
Now we take up <u>curvature</u> and <u>holonomy</u>. Consider the parallel transport of a vector $Z \in T_{x_0}M$ around a loop c based at x_0: 10/28/08



This produces a linear map: $T_{x_0}M \to T_{x_0}M : Z \mapsto Z'$ (in the picture). The map is invertible, since each infinitesimal step of the parallel transport is an invertible map between neighboring tangent spaces. Thus, the map is an element of $GL(n, \mathbb{R})$. Call the map P_c (it is parameterized by the loop c.) P_c is called the <u>holonomy</u> of the loop.

Notice in general $P_c \in GL(n,\mathbb{R})$, but if a metric exists 10/28/08
and a metric connection is employed, then P_c preserves scalar products, i.e., $P_c : T_xM \to T_xM$ is an <u>orthogonal</u> <u>transformation</u> (a member of $SO(n)$ for an orientable, Riemannian manifold, or $SO(n,m)$ for an oriented, pseudo-Riem. manifold). In general, the set of all possible holonomies of all possible loops based at x_0 is a subgroup of $GL(n,\mathbb{R})$, called the <u>holonomy group at x_0</u>, denoted $H(x_0)$. Like the fundamental group, elements of the holonomy group depend on the loop, but they are not invariant under continuous deformation. If points x_0 and x_1 can be connected by a curve as above, then $H(x_0)$ and $H(x_1)$ are conjugate groups, $H(x_0) = \tau H(x_1) \tau^{-1}$. As abstract groups they are the same. Then one can speak of the holonomy group of the manifold. For example, the holonomy group of the 2-sphere (under the Levi-Civita connection and the obvious metric) is $SO(2)$.

If the loop is <u>infinitesimal</u> then we get an infinitesimal element of the holonomy group, i.e., an element of the <u>Lie algebra</u>. E.g., consider an infinitesimal parallelogram defined by vectors X and Y:



Then the Lie algebra element you get upon parallel transporting around the small loop depends on the area element (it is linear and antisymmetric in X, Y), i.e., it is a <u>Lie algebra-valued 2-form</u>:

$$R : \underbrace{T_xM \times T_xM}_{\text{antisymm.}} \longrightarrow \text{Lie algebra of } H(x), \text{ e.g., } \begin{matrix} \mathfrak{gl}(n,\mathbb{R}) \\ \mathfrak{so}(n). \end{matrix}$$

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Conventions for attaching indices to R. Let the Lie algebra element be represented by an $n \times n$ matrix, in some basis in $T_x M$. Then write,

conventional minus sign

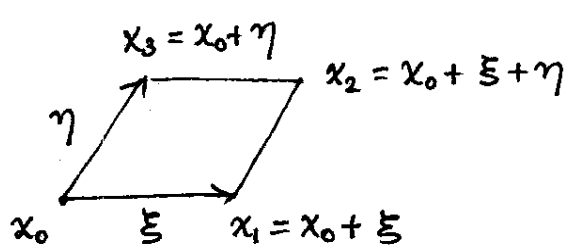
$$Z'^{\mu} = \left[\delta^{\mu}_{\nu} - R(X,Y)^{\mu}_{\cdot\,\nu}\right] Z^{\mu}$$

for the parallel transport of Z around the $X-Y$ parallelogram (along X first, then Y). The correction term is linear and antisymmetric in X, Y, hence

$$R(X,Y)^{\mu}_{\cdot,\nu} = \underbrace{R^{\mu}{}_{\nu\alpha\beta}}_{\text{curvature tensor.}} X^{\alpha} Y^{\beta}$$

where $R^{\mu}_{\cdot\,\nu\alpha\beta} = -R^{\mu}_{\cdot\,\nu\beta\alpha}$.

How to calculate $R^{\mu}_{\cdot\,\nu\alpha\beta}$ in a coordinate basis $e_{\mu} = \frac{\partial}{\partial x^{\mu}}$. change notation slightly, write ξ, η instead of X, Y (ξ, η are infinitesimals). These define an infinitesimal parallelogram in the given coordinates,



The sides of the parallelogram are straight lines in the given coordinates. Thus, on transporting a vector Z along the first leg $x_0 \to x_1$, we create a curve parametrized by t, $x^{\mu}(t) = x^{\mu}_0 + t\,\xi^{\mu}$, $0 \le t \le 1$.

Notation: Let $(\xi \cdot \Gamma)$ be the $n \times n$ matrix with components,

$$\underbrace{(\xi \cdot \Gamma)^{\mu}_{\cdot\,\nu}}_{(\Gamma_{\xi})^{\mu}_{\cdot\,\nu}} = \xi^{\alpha}\, \Gamma^{\mu}_{\alpha\nu}.$$

or Γ_{ξ}

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Equ. of parallel transport is

$$\frac{dZ^\mu}{dt} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{dt}Z^\beta .$$

But $x^\alpha(t) = x_0^\alpha + t\,\xi^\alpha$

$$\frac{dx^\alpha}{dt} = \xi^\alpha$$

so $$\frac{dZ^\mu}{dt} = -(\Gamma_\xi)^\mu{}_{\cdot\beta}\,Z^\beta,$$

$\hookrightarrow$ eval. at $x(t)$.

or $$\frac{dZ}{dt} = -\Gamma_\xi(x_0+t\xi)\,Z \qquad \text{for short.} = Z'$$

then $$\frac{d^2Z}{dt^2} = -\xi\cdot\nabla\Gamma_\xi\,Z - \Gamma_\xi\frac{dZ}{dt} = Z''.$$

$$= \left(-\xi\cdot\nabla\Gamma_\xi + \Gamma_\xi^2\right)Z \qquad \text{where } \xi\cdot\nabla\Gamma_\xi = \xi^\mu(\Gamma_\xi)_{,\mu}.$$

So, $$Z_0' = -\Gamma_\xi Z_0$$

$$Z_0'' = \left(-\xi\cdot\nabla\Gamma_\xi + \Gamma_\xi^2\right)Z_0.$$

So, $$Z_1 = \left[\mathrm{Id} - \Gamma_\xi + \tfrac{1}{2}\left(-\xi\cdot\nabla\Gamma_\xi + \Gamma_\xi^2\right)\right]Z_0$$ Taylor series at $t=1$.

everything in [ ] eval at x_0. Z_1 = value of Z, parallel transported from x_0 to x_1. To transport ~~to~~ $x_1 \to x_2$, replace $Z_0 \to Z_1 \to Z_2$, $\xi \to \eta$, $x_0 \to x_1 = x_0 + \xi$. Thus,

$$Z_2 = \left[\mathrm{Id} \underbrace{- \Gamma_\eta(x_0+\xi)}_{\hookrightarrow\ -\Gamma_\eta(x_0) - \xi\cdot\nabla\Gamma_\eta} + \tfrac{1}{2}\left(-\eta\cdot\nabla\Gamma_\eta + \Gamma_\eta^2\right)\right]Z_1$$

Summary...

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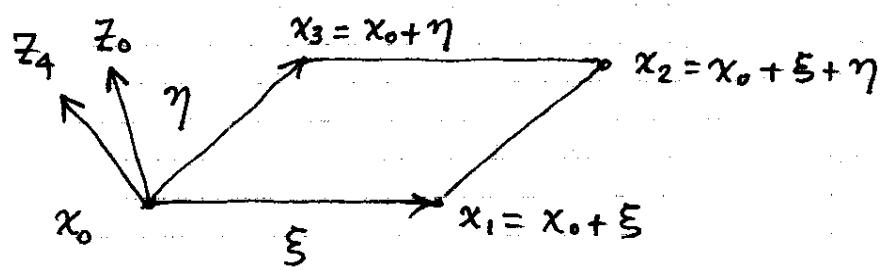
Computation of curvature tensor. Given manifold M with connection ∇, but not nec. anything else (such as g). Parallel transport a vector Z around the 4 sides of a parallelogram spanned by two infinitesimal vectors ξ, η, with corners x_0, x_1, x_2, x_3, giving $Z_0 \to Z_1 \to Z_2 \to Z_3 \to Z_4$, where $Z_0, Z_4 \in T_{x_0}M$:



The transport $Z_0 \to Z_4$ must be linear and near-identity (since the parallelogram is small). Write it

$$Z_4^\mu = \left[\mathrm{Id} - R(\xi,\eta)\right]^\mu{}_{\cdot\,\nu} Z_0^\nu$$

where $R(\xi,\eta)^\mu{}_{\cdot\,\nu}$ is linear in ξ, η, so

$$R(\xi,\eta)^\mu{}_{\cdot\,\nu} = R^\mu{}_{\cdot\,\nu\alpha\beta}\, \xi^\alpha \eta^\beta$$

defines the components $R^\mu{}_{\cdot\,\nu\alpha\beta}$ of the curvature tensor. It is anti-symmetric in ξ, η, since if $\xi = \eta$, then you are parallel transporting along a line and back again, which cancels (gives $R(\xi,\xi) = 0$). Thus we expect

$$R^\mu{}_{\cdot\,\nu\alpha\beta} = -R^\mu{}_{\cdot\,\nu\beta\alpha}.$$

Equivalently, R is a <u>Lie-algebra valued 2-form.</u> Of course we must verify these expected properties of R (such as the fact that it is a tensor).

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On the leg $x_0 \to x_1$, the $\parallel$-transport equation can be solved in a Taylor series in the small displacement ξ, which we expand through 2nd order:

not covariant deriv., just shorthand.

$$Z_1 = \left[\mathrm{Id} - \Gamma_\xi(x_0) + \tfrac{1}{2}\left(- \xi\cdot\nabla\Gamma_\xi + \Gamma_\xi^2 \right) \right] Z_0$$

shorthand for $Z_1^\mu = [\ \ ]^\mu{}_{\cdot\nu}\, Z_0^\nu$

where $(\Gamma_\xi)^\mu{}_{\cdot\nu} = \xi^\alpha \Gamma^\mu_{\alpha\nu}$

$$(\xi\cdot\nabla\Gamma_\xi)^\mu{}_{\cdot\nu} = \xi^\beta \xi^\alpha \Gamma^\mu_{\alpha\nu,\beta}.$$

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Similarly,

$$Z_3 = \left[Id + \Gamma_\xi(x_0+\xi+\eta) + \tfrac{1}{2}\left(-\xi\cdot\nabla\Gamma_\xi + \Gamma_\xi^2\right)\right] Z_2$$

$$Z_4 = \left[Id + \Gamma_\eta(x_0+\eta) + \tfrac{1}{2}\left(-\eta\cdot\nabla\Gamma_\eta + \Gamma_\eta^2\right)\right] Z_3 .$$

Now multiply matrices,

$$Z_4 = \underbrace{[\ ][\ ][\ ][\ ]}\, Z_0$$

$$= Id - \left[\Gamma_\xi\Gamma_\eta - \Gamma_\eta\Gamma_\xi + \xi\cdot\nabla\Gamma_\eta - \eta\cdot\nabla\Gamma_\xi \right]$$

$$= Id - R(\xi,\eta).$$

From this can read off components of R,

$$\boxed{R^\mu{}_{\nu\alpha\beta} = \Gamma^\mu_{\alpha\sigma}\Gamma^\sigma_{\beta\nu} - \Gamma^\mu_{\beta\sigma}\Gamma^\sigma_{\alpha\nu} + \Gamma^\mu_{\beta\nu,\alpha} - \Gamma^\mu_{\alpha\nu,\beta}}$$

Expression of curvature tensor in terms of connection, in a coordinate basis $e_\mu = \partial/\partial x^\mu$. (No metric required.)

Γ is not a tensor, but R should be a tensor, based on its definition (its a mapping of T_xM onto itself, given ξ and η). A direct proof that R transforms as a tensor is tedious, however.

A coordinate-free approach is better. We define

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),$$

notation $R(X,Y,Z) = \underbrace{R(X,Y)}Z$

↳ will turn out to be same $R(X,Y)$ defined above

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where

$$R(X,Y): \mathfrak{X}(M) \to \mathfrak{X}(M),$$

$$\boxed{R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}}$$

coordinate-free defn of R.

First show that this is a tensor. Must be linear in X, Y. (pointwise) Because of antisymmetry, suffices to check only X. Let $f \in \mathcal{F}(M)$

$$R(fX,Y) = \underbrace{\nabla_{fX}\nabla_Y}_{\to\, f\nabla_X\nabla_Y} \underbrace{- \nabla_Y \nabla_{fX}}_{\to\, -\nabla_Y f \nabla_X = -(Yf)\nabla_X - f\nabla_Y\nabla_X} \underbrace{- \nabla_{[fX,Y]}}_{\to\, = -\nabla_{f[X,Y]-(Yf)X} = -f\nabla_{[X,Y]} + (Yf)\nabla_X}$$

$$= f\,R(X,Y).$$

should also be linear in Z. Check it.

$$R(X,Y)fZ = \nabla_X\nabla_Y fZ - \nabla_Y\nabla_X fZ - \nabla_{[X,Y]} fZ$$

$$= T_1 + T_2 + T_3 \quad \text{(3 terms)}.$$