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$f \circ c$. If you think of c as a map from $\mathbb{R}^r$ to M (actually it is a linear comb. of such maps, the singular cubes or simplexes) then $f \circ c$ is a map : $\mathbb{R}^r \to N$. Then we have the following fact:

$$\boxed{\int_{f\circ c} \omega = \int_c f^* \omega .}$$

This is easily proved by resorting to the definition of the integral. This definition proceeds by pulling everything back to $\mathbb{R}^r$, so it doesn't matter if we pull it back in stages or all at once, $(f \circ c)^* = c^* \circ f^*$.

Now we turn to <u>potentials</u>. If a r form ω can be written

$$\omega = d\psi$$

for some $(r-1)$ form ψ, then we say ψ is a potential for ω. Only exact forms have (global) potentials; this is just the meaning of "exact".

The identity $dd = 0$ means that exact $\Rightarrow$ closed. But closed $\not\Rightarrow$ exact, as we see from the monopole example. At least, this is true in a global sense on most manifolds. But it turns out that closed always $\Rightarrow$ exact <u>locally</u>. This is called the <u>Poincaré lemma</u> by Nakahara. This comes in several versions that we will explore.

<u>First version</u>. Let $\omega \in Z^r(M)$, $d\omega = 0$, Then ~~in~~ for every $p \in M$ there exists a neighborhood of p such that $\exists$ an $(r-1)$-form ψ such that $\omega = d\psi$ on this neighborhood.

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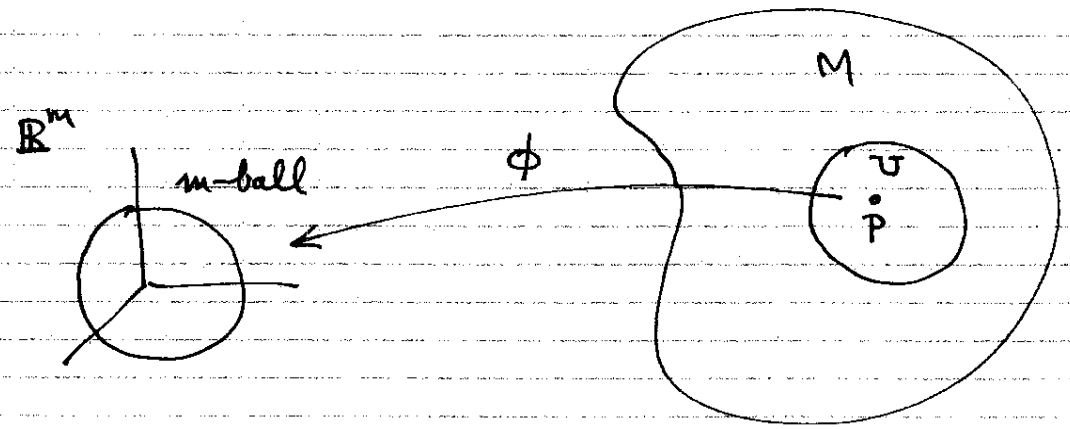
<u>Proof.</u> Every $p \in M$ possesses a neighborhood U (sufficiently small) that it is diffeomorphic to the m-ball (where $m = \dim M$). Take this as obvious.



Let $\phi: U \to$ m-ball be the diffeomorphism defining a coordinate chart, Let x^μ be the coordinates, and assume $x^\mu(p) = 0$. Let $\omega_{\mu_1 \ldots \mu_r}(x)$ be the components of ω in this chart. Define $\psi \in \Omega^{r-1}(M)$ by its components,

$$\boxed{\psi_{\mu_1 \ldots \mu_{r-1}}(x) = \int_0^1 dt\; t^{r-1}\, x^\sigma\, \omega_{\sigma \mu_1 \ldots \mu_{r-1}}(tx).}$$

This is the <u>Volterra formula</u>. From it one can prove directly that $\omega = d\psi$ inside the ball (or U). The formula is written in components, but you can see that it involves integrating ω along a straight radial path (in the given coordinates) from p to x, to get the value of ψ at x.

<u>Proof</u> of the Volterra formula for $r=2$: Change notation, $\omega \to F$, $\psi \to A$ to make it look more like E+M.

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Then the Volterra formula is

$$A_\mu(x) = \int_0^1 dt \; t \; x^\sigma F_{\sigma\mu}(tx).$$

Then calculate:

$$A_{\mu,\nu}(x) = \int_0^1 dt \left[t \underbrace{\delta^\sigma_\nu F_{\sigma\mu}(tx)}_{\to F_{\nu\mu}(tx).} + t^2 x^\sigma F_{\sigma\mu,\nu}(tx) \right]$$

$$A_{\nu,\mu}(x) = \int_0^1 dt \left[t F_{\mu\nu}(tx) + t^2 x^\sigma F_{\sigma\nu,\mu}(tx) \right]$$

$$(dA)_{\mu\nu}(x) = \int_0^1 dt \left[2t F_{\mu\nu}(tx) + t^2 x^\sigma \underbrace{(F_{\sigma\nu,\mu} + F_{\mu\sigma,\nu})}_{= -F_{\nu\mu,\sigma} = +F_{\mu\nu,\sigma}} (tx) \right]$$

since $dF=0$.

$$= \int_0^1 dt \frac{d}{dt}\left[t^2 F_{\mu\nu}(tx)\right] = F_{\mu\nu}(x). \qquad \text{QED}, \quad \underline{F = dA}.$$

Next, another version of the Poincaré lemma, this time restricted to 1-forms. A basic theorem in calculus is the following. Consider the differential equations on $\mathbb{R}^n$,

$$\frac{\partial f}{\partial x^\mu} = A_\mu(x),$$

where $A_\mu(x)$ are given functions. Then this set of equations has a solution for f on a simply connected region iff $A_{\mu,\nu} = A_{\nu,\mu}$, i.e., if $dA=0$. In other words, closed $\Rightarrow$ exact on a simply connected region of $\mathbb{R}^n$.

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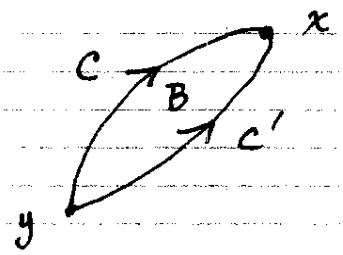
We will prove a version of this on manifolds. Let R be a simply connected region of a manifold M, let $\omega \in \Omega^1(M)$ be closed, $d\omega = 0$. Let y be a fixed pt. of R and x another such point, and let C be a curve joining y to x (confined to R). The integral $\int_C \omega$ does not depend on the path connecting y to x, because if we choose another path C', C can be continuously deformed into C' (they are homotopic, thereby sweeping out a 2-dimensional region B with boundary $\partial B = C - C'$:



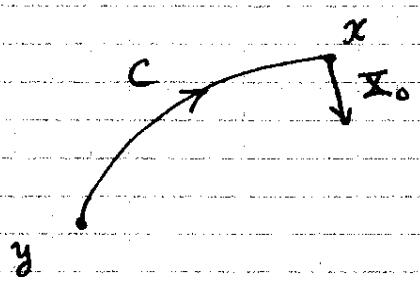
and

$$\int_C \omega - \int_{C'} \omega = \int_{\partial B} \omega = \int_B d\omega = 0.$$

So this integral defines a function

$$f(x) = \int_C \omega$$

that is, it depends only on the endpoints. Now let X_0 be a specific vector in $T_x M$,



We wish to compute $(X_0 f)$.

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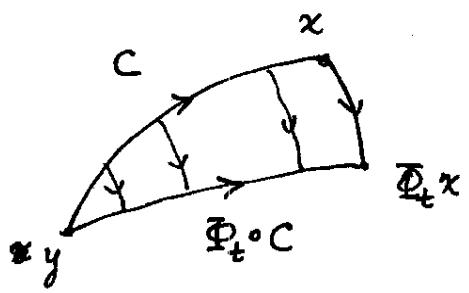
Promote X_0 into a smooth vector field such that $X = X_0$ at x, and $X = 0$ at y. Otherwise X is arbitrary. Let Φ_t be the advance map. Let each point of C flow with Φ_t to create a new curve $\Phi_t \circ C$:



The point y does not move under the flow since $X|_y = 0$. To compute $f(\Phi_t x)$ we can use any path connecting y to $\Phi_t x$. The path $\Phi_t \circ C$ is convenient. Thus,

$$f(\Phi_t x) = \int_{\Phi_t \circ C} \omega$$

or

$$(\Phi_t^* f)(x) = \int_C \Phi_t^* \omega .$$

Now apply $\frac{d}{dt}\Big|_{t=0}$, use $\frac{d}{dt}\Big|_{t=0} \Phi_t^* = \mathcal{L}_X$ when acting on forms. (the Lie derivative)

This gives

$$(\mathcal{L}_X f)(x) = \int_C \mathcal{L}_X \omega$$

or

$$(Xf)(x) = df(X)\big|_x = \int_C \left(i_X \cancel{d\omega} + d\, i_X \omega\right)$$

use Cartan formula. since $d\omega = 0$

$$= \int_{\partial C} i_X \omega = \omega(X)\Big|_y^x = \omega(X)\big|_x .$$

So, $df(x)|_x = \omega(x)|_x$, so $df(X_0) = \omega(X_0)$ so $\boxed{df = \omega}$ since x and X_0 were arbitrary. QED.

This result is interesting because it mixes homotopy and cohomology. It is discussed in different form in the book.

Actually it is very easy to understand this result from another standpoint, using the machinery discussed so far in the course. For simplicity suppose M is simply connected (instead of some region of M). Then $\pi_1(M) = \{0\}$. But we know that $H_1(M)$ is $\pi_1(M)$ divided by the commutator subgroup. But since $\pi_1(M) = \{0\}$, $H_1(M) = \{0\}$, too. This means that all closed 1-forms are exact. Note that the condition that M be simply connected is actually too strong; we could have a nontrivial $\pi_1(M)$, as long as all 1-cycles were boundaries ($H_1(M) = \{0\}$).

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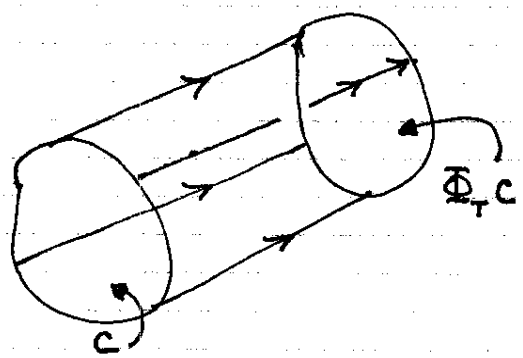
Earlier we promised a geometrical interpretation of the Cartan formula,

$$L_X = i_X d + d i_X$$

(when acting on forms). Our proof of this formula was nongeometrical (in a HW). It turns out that the Cartan formula is closely related to the proof of the Poincaré lemma.

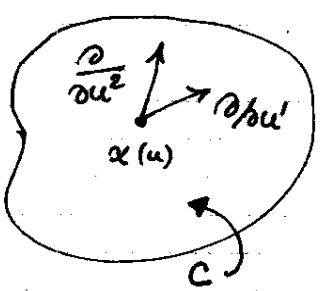
Let $\omega \in \Omega^r(M)$, $c \in C_r(M)$. We will draw the chain c as if it is a piece of an r-dimensional surface, but everything goes through for actual chains. Let $X \in \mathfrak{X}(M)$ be a vector field on M with advance map Φ_t. Let the chain c flow with the flow, sweeping out an $(r+1)$-chain, like a "tube". After time $t = T$:



We can integrate $d\omega$ over the tube and use Stokes' theorem:

$$\int_{\text{tube}} d\omega = \int_{\Phi_T c} \omega - \int_c \omega + \int_{\text{walls}} \omega .$$

Let $(u^1, \dots, u^r)$ be coordinates on c,



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so that

$$\int_c \omega = \int du^1 \ldots du^r \; \omega\Big|_{x(u)} \left(\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^r}\right).$$

where $x(u)$ ~~are~~ is the mapping : $\mathbb{R}^r \to M$ giving coordinates on c (or, rather, the mapping defining the r-chain). As for the $(r+1)$-chain "tube" swept out by c, coordinates on it are $(t, u^1, \ldots, u^r)$, where $0 \le t \le T$. ~~Thus~~ These coordinates correspond to a point

$$(t, u^1, \ldots, u^r) \longmapsto \Phi_t\, x(u).$$

So,

$$\int_{tube} d\omega = \int_0^T dt \int du^1 \ldots du^r \; \underbrace{(d\omega)\Big|_{\Phi_t x(u)} \left(X, \Phi_{t*}\frac{\partial}{\partial u^1}, \ldots, \Phi_{t*}\frac{\partial}{\partial u^r}\right)}:$$

$$= (i_X d\omega)\Big|_{\Phi_t x(u)} \left(\Phi_{t*}\frac{\partial}{\partial u^1}, \ldots, \Phi_{t*}\frac{\partial}{\partial u^r}\right)$$

$$= \left(\Phi_t^* \, i_X d\omega\right)\Big|_{x(u)} \left(\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^r}\right).$$

So,

$$\int_{tube} d\omega = \int_0^T dt \int_c \Phi_t^* \, i_X d\omega .$$

Similarly, let $(v^1, \ldots, v^{r-1})$ be coordinates for ∂c, so that $(t, v^1, \ldots, v^{r-1})$ are coordinates for the walls. Then

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$$\int_{\text{walls}} \omega = -\int_0^T dt \int_{\partial C} \Phi_t^* i_X \omega = -\int_0^T dt \int_C d\, \Phi_t^* i_X \omega .$$

The minus sign takes some attention to orientation rules.

Finally,

$$\int_{\Phi_T C} \omega = \int_C \Phi_T^* \omega ,$$

using property of integrals under maps. Altogether, we have

$$\boxed{\int_0^T dt \int_C \Phi_t^* i_X d\omega = \int_C \Phi_T^* \omega - \int_C \omega - \int_0^T dt \int_C d\, \Phi_t^* i_X \omega .}$$

Now apply $\frac{d}{dT}\Big|_{T=0}$, use $\frac{d}{dT}\Big|_{T=0} \Phi_T^* \omega = L_X \omega$, $\Phi_0^* = \text{id}$.

Find,

$$\int_C i_X d\omega = \int_C L_X \omega - 0 - \int_C d i_X \omega ,$$

or, since C is arbitrary,

$$L_X \omega = i_X d\omega + d i_X \omega .$$

The Cartan formula arises when surfaces or chains are allowed to flow under some advance map of a vector field. It arises when we apply $\frac{d}{dt}$ to the integral of ω over a chain being transported by flow. Like a small disk.

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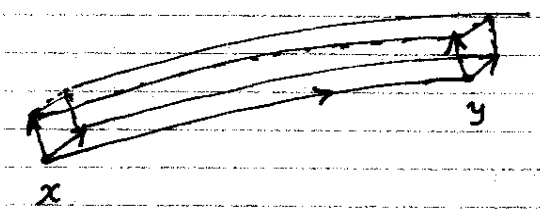
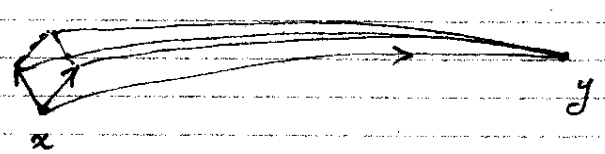
Let's apply the boxed formula in another extreme. Suppose the surface c is ~~sp~~ a small parallelepiped, spanned by r vectors at a point $x \in M$. This surface is allowed to flow under Φ_T to a point y.



Allowing the small parallelogram to flow means the vectors are mapped by Φ_{t*}. Suppose also that $d\omega = 0$. Then boxed formula implies,

$$\int_{\Phi_T c} \omega = \int_c \omega + \int_0^T dt \int_c d\, \Phi_t^* \, i_x \omega .$$

Now suppose the flow generates a deformation retract to the point y. Then it looks like this:



and all transported vectors collapse to 0 when they reach y. Thus

$$\int_{\Phi_T c} \omega = 0, \quad \text{and}$$

$$\int_c \omega = -d \int_c \int_0^T dt \;\; \Phi_t^* \, i_x \omega .$$

But since c is arbitrary, this implies

$$\omega = -d \int_0^T dt \;\; \Phi_t^* \, i_x \omega$$

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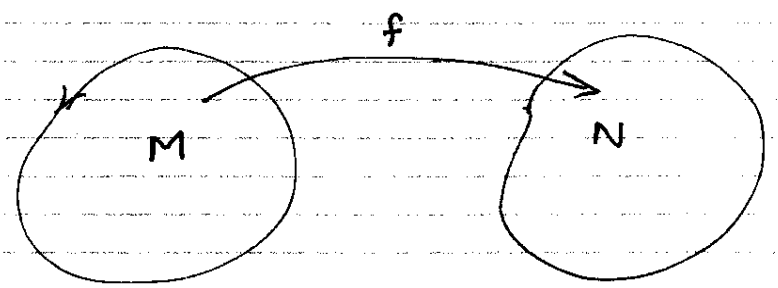
Thus we see that on a contractible region, every closed form is exact (another version of the Poincaré lemma). If you make the contraction flow along radial lines in some coordinates,

$$\frac{dx^\mu}{dt} = -x^\mu, \qquad x^\mu(t) = x_0^\mu e^{-t},$$

then (with $T \to \infty$) the result above implies the Volterra formula.

Note: Since $\mathbb{R}^n$ is contractible, closed $\Leftrightarrow$ exact on $\mathbb{R}^n$.

A third application of the boxed formula concerns the behavior of cohomology groups under maps. Let $f: M \to N$ be a smooth map.



We know f^* can be used to pull-back forms on N to forms on M. Can this also be used to pull back elements of cohomology groups? Let $\omega \in \Omega^r(N)$, $d\omega = 0$. Then $[\omega] \in H^r(N)$. The obvious definition is

$$f^*[\omega] = [f^*\omega],$$

but we have to check that this makes sense. First, $df^*\omega = f^*d\omega = 0$, so $f^*\omega$ is closed on M. Next, if $\omega' = \omega + d\psi$, then $f^*\omega' = f^*\omega + f^*d\psi = f^*\omega + d(f^*\psi)$. So, $[f^*\omega] = [f^*\omega']$, and the answer (the pull-back of $[\omega]$) does not depend on which representative element we choose in $[\omega]$.

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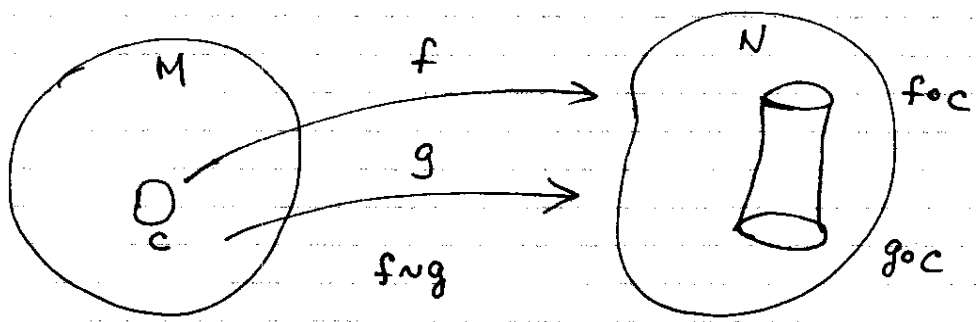
Now let f, g be two maps: $M \to N$. Suppose they are <u>homotopic</u> ($f \sim g$). (We explore some connections between homotopy and cohomology.). Let c be an r-chain in M, gives 2 r-chains $f \circ c$ and $g \circ c$ in N, that can be deformed into one another by the homotopy that deforms f into g.



Now let $[\omega] \in H^r(N)$, so $d\omega = 0$, and integrate $d\omega$ over the volume of the tube in N, using the boxed formula. Assume that the deformation given by the homotopy is the advance map of a flow associated with a vector field X on N. Then we find (replacing c in the boxed formula with $f \circ c$)

$$0 = \int_{g \circ c} \omega - \int_{f \circ c} \omega - d \int_{f \circ c} \int_0^1 dt \; \Phi_t^* i_X \omega$$

or
$$0 = \int_c g^* \omega - \int_c f^* \omega - d \int_c \int_0^1 dt \; f^* \Phi_t^* i_X \omega$$

or since c is arbitrary,

$$g^* \omega = f^* \omega + d\psi, \qquad \psi = - \int_0^1 dt \; f^* \Phi_t^* i_X \omega$$

or
$$[g^* \omega] = [f^* \omega]$$

or
$$g^* [\omega] = f^* [\omega].$$

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So pull-backs of cohomology groups under homotopic maps are identical,

$$f^* H^r(N) = g^* H^r(N) \quad \text{if} \quad f \sim g.$$

In the applications above of the boxed formula, p.3, it was assumed that the deformation associated with the homotopy could be realized as an advance map of some vector field. This was done only for reasons of laziness. You can fix this up, and the answers still hold.

Nakahara discusses "Poincaré duality," but his arguments cannot be understood on the basis of material covered so far in the course. So I will just quote the result. Let M be compact, so the $H^r(M)$ are finite dimensional. Let $\dim M = m$. Then

$$\dim H^r(M) = \dim H^{m-r}(M),$$

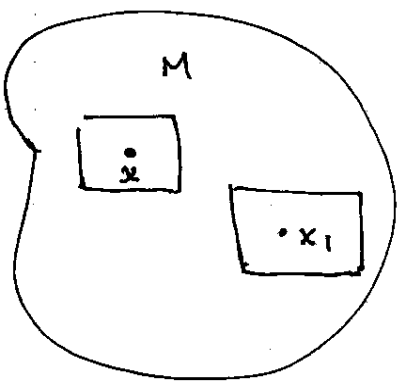
$$\text{or} \qquad b_r(M) = b_{m-r}(M) \qquad \text{(Betti numbers).}$$

The proof of this requires harmonic forms, which we consider later.

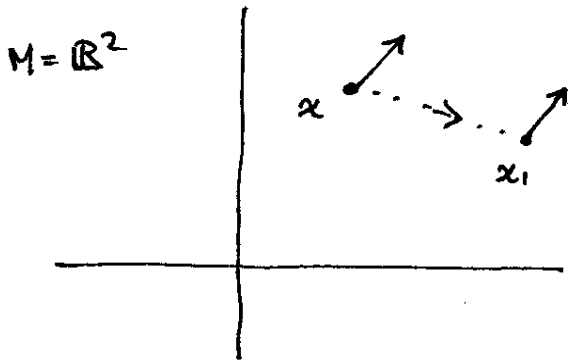
He also discusses the Künneth formula, which concerns the cohomology of Cartesian product spaces. We'll come back to this later if we need it.

Now we turn to a new type of geometrical structure, a <u>connection</u>. Connections are often discussed in the context of a metric, but the two ideas are actually distinct, and you don't need a metric to have a connection. For example, connections are sometimes important on symplectic manifolds.

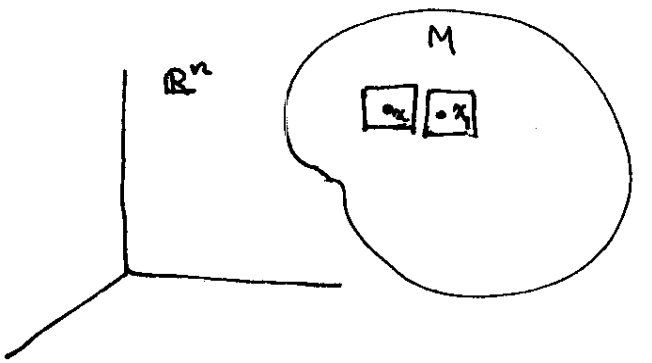
In general, on a manifold there is no <u>natural</u> way to identify two tangent spaces at two different points x, x_1, even though both are n-dimensional ($n = \dim M$). To create such an identification, it is necessary to introduce some additional geometrical structure. An <u>exception</u> is the case that $M =$ a vector space $= \mathbb{R}^n$, then all tangent spaces can be identified with one another (and with $M = \mathbb{R}^n$ itself).



You just use the vector space structure to move a vector based at one point parallel to itself over to another point.



Another example to think about is the case of a submanifold M ($\dim M = m$) of Euclidean $\mathbb{R}^n$ ($n \geq m$). For now we speak intuitively (use infinitesimals, etc.)



In this case you can use the geometry (vector space + Euclidean) of $\mathbb{R}^n$ to create an isomorphism between tangent space at nearby (infinitesimally separated) points x and x_1 on M. The rule is the following. ①. Take a vector $Y \in T_x M$ and move it parallel to itself (in $\mathbb{R}^n$) over to the nearby point $x_1 = x + \Delta x$. Unfortunately, this transported vector is not tangent to M at x_1 (it belongs to $T_{x_1}\mathbb{R}^n$, but not $T_{x_1}M$). To fix this, ② use the metric in $\mathbb{R}^n$ to project the vector onto the tangent plane at x_1, producing $Y' \in T_{x_1}M$.

You <u>could</u> use this ~~definition~~ construction to identify tangent spaces to M at remote points, but it would not be a useful definition. Instead, it's best to identify distant tangent spaces by linking them by a chain of infinitesimal increments. Then you find that the identification is

path-dependent. So for now concentrate on the identification 10/23/08 of nearby tangent spaces only.

In general, we want a map,

$$: T_x M \to T_{x+\Delta x} M : Y \mapsto Y', \text{ say, when } \Delta x \text{ small,}$$

such that Y' is linear in Y. Notice that we cannot say, we want the mapping to be close to the identity when Δx is small, since we have no way of defining the identity map between $T_x M$ and $T_{x+\Delta x} M$. But if we impose coordinates $\{x^\mu\}$ and use the coordinate basis $\{\partial/\partial x^\mu\}$ to compute components, then the components Y'^μ should be close to Y^μ, and the correction should be linear in Δx. Thus we want

$$Y'^\mu = Y^\mu + (\text{something linear in } \Delta x, Y)$$

$$= Y^\mu - \Gamma^\mu_{\alpha\beta} \Delta x^\alpha Y^\beta .$$

The minus sign is conventional. $\Gamma^\mu_{\alpha\beta}$ are the <u>connection coefficients</u>, they are the coefficients of the linear relationship.

In a sense, Y' is the "same" vector at $x+\Delta x$ as Y was at x. More precisely, Y' is the <u>parallel transported</u> version of Y. Here we have just transported a specific vector Y <u>at the point</u> x over to a nearby point. But if Y is a vector <u>field</u>, then it has a value at $x+\Delta x$, and in general, $Y(x+\Delta x) \neq Y'$. But since

$$Y^\mu(x+\Delta x) = Y^\mu(x) + \Delta x^\alpha \underbrace{Y^\mu_{,\alpha}}_{\text{called simply } Y \text{ above.}}$$

then the difference is $\quad Y^\mu(x+\Delta x) - Y'^\mu = \Delta x^\alpha (Y^\mu_{,\alpha} + \Gamma^\mu_{\alpha\beta} Y^\beta).$

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Now replace Δx^α by εX^α, where X is some vector in $T_x M$ (representing the displacement), and define the <u>covariant derivative</u> of Y along X to be

$$(\nabla_X Y)^\mu = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[Y^\mu(x+\varepsilon X) - Y'^\mu \right]$$

$$= X^\alpha \left(Y^\mu_{,\alpha} + \Gamma^\mu_{\alpha\beta} Y^\beta \right).$$

In this construction, Y must be a <u>vector field</u>, while X need only be defined at one point x (of course, it may be a field, too).

The covariant derivative gives a notion of the <u>directional derivative</u> of a vector field Y along X, although it depends on a connection. The Lie derivative does not do this, since the Lie derivative $\mathcal{L}_X Y$ involves derivatives of X as well as Y.

Now we make an <u>abstract</u> <u>approach</u> to covariant derivatives. We wish to define,

$$\nabla : \; T_x M \times \mathfrak{X}(M) \to T_x M \qquad (X \text{ a vector at a pt})$$

$$\text{or} \quad \nabla : \; \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \qquad (X \text{ a vector field})$$

such that:

$$\left.\begin{array}{ll} 1. & \nabla_{fX} Y = f \nabla_X Y \\ 2. & \nabla_{(X+Y)} Z = \nabla_X Z + \nabla_Y Z \\ 3. & \nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z \\ 4. & \nabla_X (fY) = (Xf) Y + f \nabla_X Y \end{array}\right\} \text{ these properties are all satisfied by our definition above.}$$

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Put this into a basis. For now use only a coordinate basis, so that $e_\mu = \partial/\partial x^\mu$. Define connection components by

$$\nabla e_\alpha \equiv \nabla_\alpha$$

$$\nabla_\alpha e_\beta = \text{a vector} = e_\gamma \, \Gamma^\gamma_{\alpha\beta}.$$

Now use rules above to compute $\nabla_X Y$ in components. Write $X = X^\mu e_\mu$, $Y = Y^\mu e_\mu$. Then

$$\begin{aligned}
\nabla_X Y &= \nabla_{X^\mu e_\mu}(Y^\nu e_\nu) \\
&= X^\mu \nabla_\mu (Y^\nu e_\nu) \\
&= X^\mu \left[(\nabla_\mu Y^\nu)\, e_\nu + Y^\nu \nabla_\mu e_\nu \right] \\
&= X^\mu \left[Y^\nu_{,\mu}\, e_\nu + Y^\nu e_\alpha \Gamma^\alpha_{\mu\nu} \right] \\
&= X^\mu \left[Y^\alpha_{,\mu} + Y^\nu \Gamma^\alpha_{\mu\nu} \right] e_\alpha .
\end{aligned}$$

Agrees with earlier calculation of $(\nabla_X Y)^\mu$. Shows or indicates equivalence of two points of view.

<u>Parallel transport</u>. Now that we know how to parallel transport a vector over an infinitesimal segment, by integration we can transport over a finite distance, along a curve. Let x_0 be the beginning of a curve, let $Y_0 \in T_{x_0}M$ be a given tangent vector at the initial point. Y_0 need only be defined at one point, i.e., it need not be a field. Let s be a parameter of the curve (any parameter).