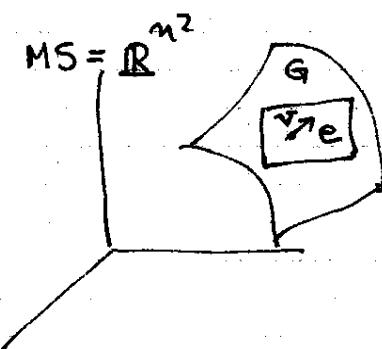


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Now we consider how things like the Lie algebra, one-parameter subgroups, etc., are expressed in terms of matrices in the case that we have a matrix group. Consider a real matrix group, for simplicity. As explained previously, such a group can be thought of as a submanifold of "matrix space" $MS = \mathbb{R}^{n^2}$, where our group consists of real, $n \times n$ matrices.



Then every point of G is also a matrix, and matrix multiplication and inversion ~~g~~ correspond to group multiplication and inversion. (and $e=I$)

We will not attempt to put coordinates on G , but coordinates on MS may be taken to be the components of a matrix. That is, if $M \in MS$, let us write

$$M = \begin{pmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix}$$

so that $\{x_{ij}\}$ are the coordinates on MS .

A vector at a point to G can also be interpreted as a vector at the same point to MS , and thus can be expanded as a linear combination of the basis vectors $\partial/\partial x_{ij}$. For example, if $v \in g$, then we can write

$$v = \sum_{ij} v_{ij} \frac{\partial}{\partial x_{ij}},$$

where v_{ij} is a matrix. This is the usual matrix belonging to the Lie

algebra of a matrix group. For example, if $G = \text{SO}(3)$, $MG = \mathbb{R}^9$, then the Lie algebra consists of antisymmetric matrices. Then, $V_{ij} = -V_{ji}$.

Then it also happens that the one-parameter subgroups $\exp(tV)$ defined in the differential-geometric setting coincide with matrix exponentiation $\exp(tV)$ (same notation). It also happens that the $[,]$ bracket on the Lie algebra becomes the ordinary matrix commutator. Other objects (left and right translations, left-invariant vector fields, etc) can also be translated into matrix language.

Return to the differential geometry of Lie groups. Let $\{V_\mu, \mu=1, \dots, n\}$ ($n = \dim V$) be a basis in \mathfrak{g} . Let $X_\mu = X_{V_\mu}$ be the corresponding LIVF's. Now the bracket $[V_\mu, V_\nu]$ is also a vector in \mathfrak{g} , so it can be expanded in terms of the $\{V_\mu\}$,

$$[V_\mu, V_\nu] = C_{\mu\nu}^\sigma V_\sigma,$$

where $C_{\mu\nu}^\sigma$ are the expansion coefficients. These numbers are called the structure constants of the Lie algebra, although they are not really constant, instead they are the components of a type $(1, 2)$ tensor at $e \in G$. (They depend on the basis.) If we left-translate the above, we get

$$[X_\mu, X_\nu] = C_{\mu\nu}^\sigma X_\sigma,$$

with the same $C_{\mu\nu}^\sigma$ (which do not depend on position).

A different point of view results from shifting attention from vector fields to forms (the dual point of view). Let $\mathfrak{g}^* = T_e^*G$ be the dual of \mathfrak{g} (the Lie algebra). Let $\{\beta^\mu, \mu=1, \dots, n\}$ be the basis in \mathfrak{g}^* dual to $\{V_\mu, \mu=1, \dots, n\}$, the (some) given basis in \mathfrak{g} .

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That is, $\beta^\mu \in \mathcal{E}^*$,

$$\beta^\mu(v_\nu) = \delta_\nu^\mu.$$

Then define a 1-form $\theta^\mu \in \mathcal{X}^*(G)$ by

$$\theta^\mu|_a = L_{a^{-1}}^* \beta^\mu.$$

(The difference betw. β^μ and θ^μ is that β^μ is a covector at one point $a \in G$, whereas θ^μ is a covector field, i.e., a 1-form. It is like the difference between $v_\mu \in \mathcal{E}$ and $x_\mu \in \mathcal{X}(G)$.) The forms θ^μ are left-invariant 1-forms on G . The set $\{\theta^\mu\}$ is dual to $\{x_\mu\}$ at each point $a \in G$, as we see by using the definitions,

$$\begin{aligned} \theta^\mu(\bar{x}_\nu)|_a &= \theta^\mu|_a(\bar{x}_\nu|_a) = (L_{a^{-1}}^* \beta^\mu)(L_a*_a v_\nu) \\ &= \beta^\mu(L_{a^{-1}}*_a L_a*_a v_\nu) = \beta^\mu(v_\nu) = \delta_\nu^\mu. \end{aligned}$$

Thus we have bases $\{\bar{x}_\mu\}$ and $\{\theta^\mu\}$ of vectors and 1-forms at each point of G . These are generally non-coordinate bases (see HW). It is of interest to compute the components of $d\theta^\mu$ in this basis.

$$(d\theta^\mu)(\bar{x}_\nu, \bar{x}_\sigma) = \underbrace{\bar{x}_\nu \theta^\mu(\bar{x}_\sigma)} - \underbrace{\bar{x}_\sigma \theta^\mu(\bar{x}_\nu)} - \theta^\mu([\bar{x}_\nu, \bar{x}_\sigma])$$

$$\hookrightarrow = \sum_\nu \delta_\sigma^\mu = 0 \quad \hookrightarrow \text{also } = 0$$

$$= -\theta^\mu(C_{\nu\sigma}^\tau \bar{x}_\tau) = -C_{\nu\sigma}^\mu.$$

So the structure constants (with a - sign) are the components of $d\theta^\mu$ in the basis of LIEF's $\{\bar{x}_\mu\}$. The 2-form $d\theta^\mu$ (in the abstract,

for a fixed value of μ) is

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$$d\theta^\mu = -\frac{1}{2} C_{\nu\sigma}^\mu \theta^\nu \wedge \theta^\sigma$$

Maurer-Cartan structure equations.

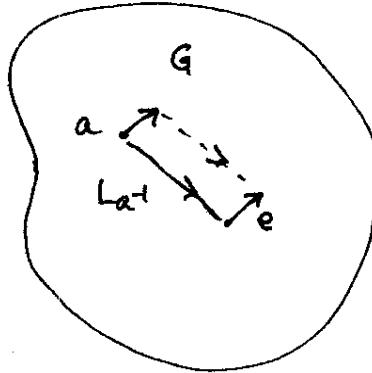
To put things in completely coordinate independent language, we write

$$\theta = V_\mu \otimes \theta^\mu.$$

θ is an example of a Lie-algebra-valued 1-form. So far we have only seen real-valued 1-forms, ~~that is,~~ but Lie-alg. valued 1-forms are important in gauge theories (gauge potentials are such things). θ is a map. (at a point $a \in G$)

$$\theta_a : T_a G \rightarrow \mathfrak{g}.$$

It is easy to see abstractly what θ does: it uses left translation to map a vector in $T_a G$ to one in $T_e G = \mathfrak{g}$.



θ is called the Maurer-Cartan form. One might say that every ^{Lie} group carries on itself a gauge potential. The MC form can be written in fully coordinate-free notation if we define

$$d\theta = d(V_\mu \otimes \theta^\mu) = V_\mu \otimes d\theta^\mu,$$

logical since the V_μ are constant. This makes $d\theta$ a lie-algebra-valued

2-form. Also define

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$$[\theta, \theta] = [v_\mu, v_\nu] \otimes \theta^\mu \wedge \theta^\nu,$$

another g -valued 2-form. Then the MC structure equations can be written,

$$d\theta + \frac{1}{2} [\theta, \theta] = 0.$$

Note, in QCD you get lie-algebra valued 1-forms, this is the gauge potential, A_μ^a where $\mu = 0, \dots, 3$ is a space-time index and $a = 1, \dots, 8$ is an index of the basis in the $SU(3)$ Lie algebra, e.g., the index of the Gell-Mann matrices. Call these V_a . Then

$$V_a A_\mu^a dx^\mu$$

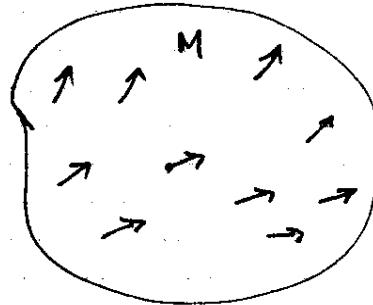
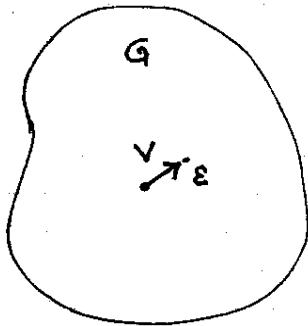
is a g -valued 1-form on space-time. (~~(And, F is)~~)

(And, $F = \frac{1}{2} V_a F_{\mu\nu}^a dx^\mu \wedge dx^\nu$ is the Yang-Mills field tensor.)

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Next we consider induced vector fields, which you have when you have an action of a Lie group on a manifold M . First the intuitive picture. Consider a vector $V \in \mathfrak{g}$. Intuitively its base is the identity e and its tip is a nearby (near-identity) group element, call it ε . The map $\Phi_\varepsilon = \text{id}_M$ does



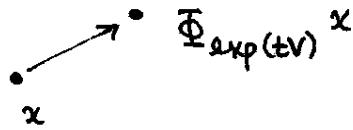
a single nothing to points of M , but Φ_ε causes the points of M to get up and move a short distance, creating a vector field on M . In this way we associate $V \in \mathfrak{g}$ with a vector field $V_M \in \mathbb{X}(M)$. (V_M is denoted V^* by Nakahara.)

To make this more precise, ~~we~~ replace ε by $\sigma(t) = \exp(tV) e = \exp(tV)$ for small t , using earlier notation for integral curves on the group manifold, and consider the action of Φ

To make this more precise we need to talk about advance maps on the group manifold, earlier denoted $\Phi_{V,t}$, and the action of G on M , which will be denoted $g \mapsto \Phi_g$. To avoid confusion, let's use $\Psi_{V,t}$ for the advance map on G , Φ_g for the action of G on M . When t is small, $\Psi_{V,t} e = \exp(tV) e$ is close to e , so we can identify it with ε above in the picture. When acting on $x \in M$, $\Phi_\varepsilon = \Phi_{\exp(tV)}$ causes x to move to a nearby point,

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thereby making a small vector on M . Letting this vector act on a ^{scalar} function $f: M \rightarrow \mathbb{R}$, we have

$$\begin{aligned} & f(\Phi_{\exp(tv)}^* x) - f(x) \\ &= (\Phi_{\exp(tv)}^* f)(x) - f(x) \\ &= ((\Phi_{\exp(tv)}^* - 1)f)(x). \quad (\text{think } t \text{ small}). \end{aligned}$$

Suggests we define $\nabla_M \in \mathcal{F}(M)$ by

$$(\nabla_M f)(x) = \left(\left. \frac{d}{dt} \right|_{t=0} \Phi_{\exp(tv)}^* f \right)(x),$$

or, since x and f are arbitrary,

$$\boxed{\nabla_M = \left. \frac{d}{dt} \right|_{t=0} \Phi_{\exp(tv)}^*}$$

(Both sides understood as operators: $\mathcal{F}(M) \rightarrow \mathcal{F}(M)$.)

See Nakahara Eq. 5.160. He drops the star on Φ and just writes g instead of Φg , where here $g = \exp(tv)$.

∇_M is called the induced vector field. It is also called the infinitesimal generator of the action $g \mapsto \Phi_g$.

An equivalent way to define the induced vector field. ∇_M eval. 10/16/08
 at a point $x \in M$ is an equivalence class of curves. One of these
 curves is easy to write down. Let $c: \mathbb{R} \rightarrow M$ be defined by

$$c(t) = \Phi_{\exp(tv)} x.$$

Then $c(0) = x$, and $[c] = \nabla_M|_x$.

An application of induced vector fields. Let M be the configuration space of a mechanical system. Impose a chart with coordinates q^μ . The Lagrangian is a function $L(q, \dot{q})$. Let G be a group with an action $g \mapsto \Phi_g$ on M , and suppose that L is invariant under the group action. This means that $\Phi_g^* L = L$, $\forall g \in G$. (But we won't define what Φ_g^* means here, just say that there is an obvious definition.) Then for every $v \in \mathfrak{g}$ there is a conserved quantity C_v , where

$$C_v = (p_\mu, \nabla_M) = p_\mu (\nabla_M)^\mu, \quad \frac{dC_v}{dt} = 0.$$

Here $p_\mu = \frac{\partial L}{\partial \dot{q}^\mu}$ is the canonical momentum. This is Noether's theorem.

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Last topic is the adjoint representation (confusingly called the "adjoint map" by Nakahara). This is a linear action of G on its own Lie algebra, $g \mapsto \text{Ad}_g$, where $\text{Ad}_g: g \rightarrow g$. The definition is simply ~~$\text{Ad}_g = L_g R_{g^{-1}}$~~

$$\text{Ad}_a = \text{I}_{a^*}|_e \text{ eval. at } e,$$

where $\text{I}_a = \text{L}_a \text{R}_{a^{-1}}$ (the inner automorphism). Thus, $\text{I}_{a^*} = \text{L}_{a^*} \text{R}_{a^{-1}*} = \text{R}_{a^{-1}*} \text{L}_{a^*}$ (since left and right translations commute). When I_{a^*} acts on a vector $v \in T_e G = g$, first L_{a^*} maps it to a vector in $T_a G$, then $\text{R}_{a^{-1}*}$ maps it back to g . So, I_{a^*} ~~also~~ maps $g \rightarrow g$. It also satisfies $\text{I}_{a^*} \text{I}_{b^*} = (\text{I}_a \text{I}_b)^* = \text{I}_{ab^*}$, since $a \mapsto \text{I}_a$ is an action. Thus, (changing notation),

$$\text{Ad}_a \text{Ad}_b = \text{Ad}_{ab}.$$

For a matrix group, a vector $v \in g$ is represented by a matrix V , group element a is rep'd by a matrix A , and $\text{Ad}_a V$ is rep'd by the matrix $A V A^{-1}$. This is the adjoint rep. for matrix groups.

(Go to p.3, 10/9/08 for notes on integrating m-forms over an m-dimensional manifold.)

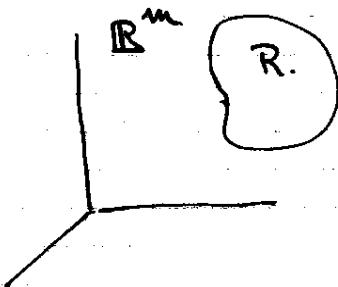
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We will take the following approach to integrating differential forms:

- (1) Integrating an m -form over an m -dimensional region of \mathbb{R}^m .
- (2) Integrating an m -form over an m -dimensional, orientable manifold M . orientable
- (3) Integrating an s -form over an s -dimensional submanifold of M ($s \leq m$).
- (4) Integrating an r -form over an r -chain.

Step 1. Let $\omega \in \Omega^m(\mathbb{R}^m)$, and let R be a "nice" region of \mathbb{R}^m .



Then ω has only one independent component ρ , given by

$$\omega = \rho(x^1, \dots, x^m) dx^1 \wedge \dots \wedge dx^m$$

That is, we use the standard coordinates on \mathbb{R}^m to define ρ .

Then we define

$$\int_R \omega = \int_R \rho(x^1, \dots, x^m) dx^1 dx^2 \dots dx^m$$

where the latter integral is an ordinary Riemann integral.

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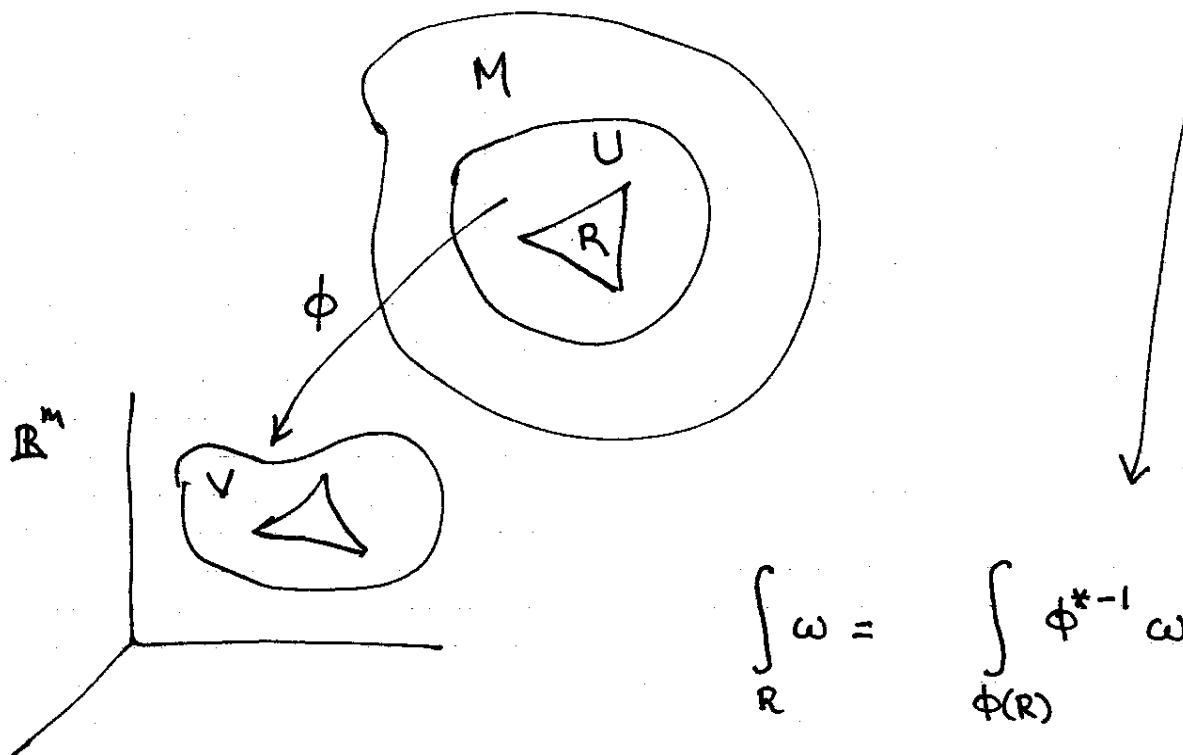
(in particular, the integral does not depend on the ordering of the dx 's.)

Step 2. Let M be an orientable, m -dimensional manifold, and let $\omega \in \Omega^m(M)$. We choose an oriented atlas on M , and divide M into regions $\{R_i\}$ such that each region R_i lies in one chart with coordinates x^k . Then we define

~~Step 2. M~~

$$\int_M \omega = \sum_i \int_{R_i} \omega,$$

where the integral over one region R is given by



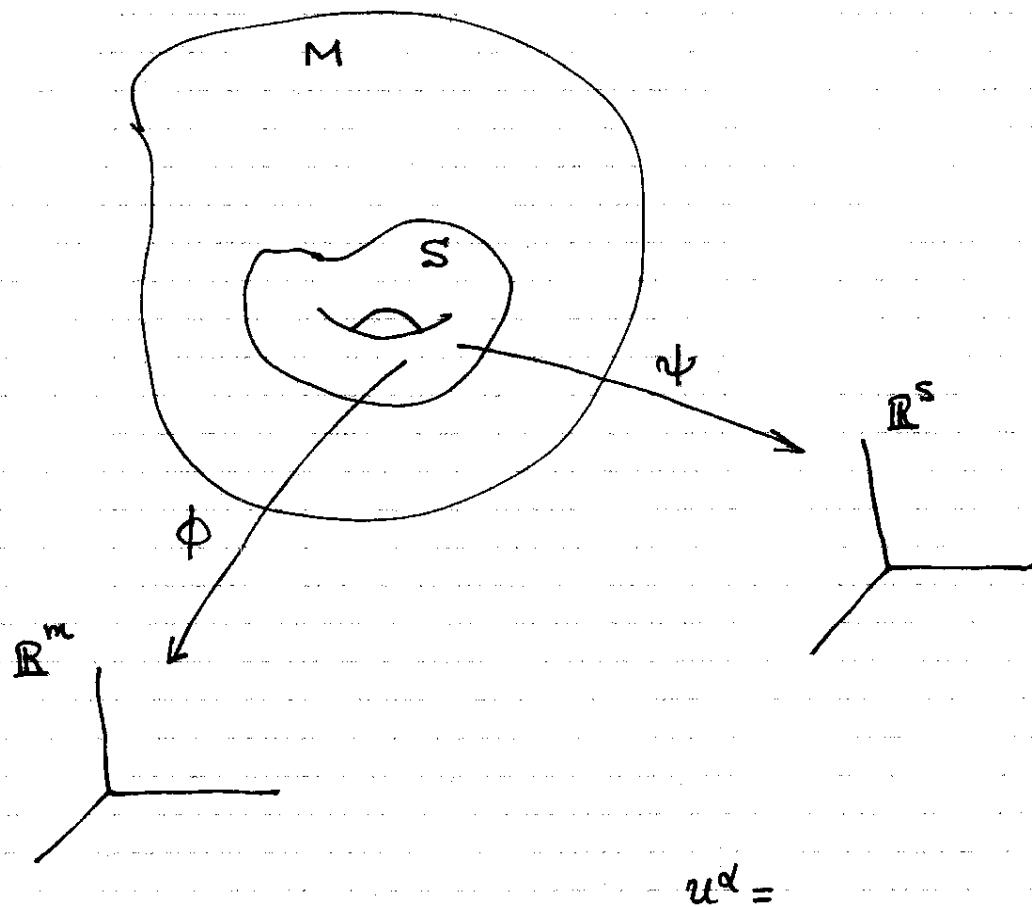
$$\int_R \omega = \int_{\phi(R)} \phi^{*-1} \omega.$$

The latter formula is the integral of an m -form over \mathbb{R}^m , which was defined in Step 1. (ϕ is the invertible coordinate map, $\phi: U \rightarrow V \subset \mathbb{R}^m$.)

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Step 3. Let S be an orientable, s -dimensional submanifold of an m -dimensional manifold M . ($s \leq m$). A submanifold of a manifold is a subset that is also a manifold. Let $\omega \in \Omega^s(M)$. Note that ω has $\binom{m}{s}$ indep. components in some chart.



since S is a manifold, we impose coordinates (u^1, \dots, u^s) on it with some chart ψ . We let this chart have some overlap with the chart ϕ on M , with coordinates $x^M = (x^1, \dots, x^m)$. In ordinary language, we would say that the functions,

$$x^M = x^M(u^\alpha) = x^M(u^1, \dots, u^s)$$

are the "equations of the surface". They represent the map

$$\phi \circ \psi^{-1} : \left(\text{region of } \frac{R^s}{R^s} \right) \rightarrow R^m$$

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Now $\omega \in \Omega^s(M)$. But a form on a manifold can always be restricted to a submanifold. In the present case, we denote the restricted form $\omega|_S$. It acts on tangent vectors to S at a point $x \in S$ by

$$(\omega|_S)|_x (x_1, \dots, x_s) = \omega|_x (x_1, \dots, x_s),$$

where $x_1, \dots, x_s \in T_x S \subset T_x M$. The vectors x_1, \dots, x_s have a dual interpretation, as vectors tangent to M , and as tangent to S . This can also be written,

$$\omega|_S = i^* \omega,$$

where $i: S \rightarrow M$ is the inclusion map (identity map on S regarded as subset of M).

Notice that it is not possible, in general, to restrict vectors, only forms.

Finally, we define

$$\int_S \omega = \int_S \omega|_S,$$

which reduces ~~one~~ Step 3 to Step 2.

Note: the final integral gets reduced (as in Step 2) to integrals over the s -dimensional coordinate space. It is interesting to write one of these integrals out in terms of the components $\omega_{\mu_1 \dots \mu_s}$ of ω on M (in chart x^{μ} on M).

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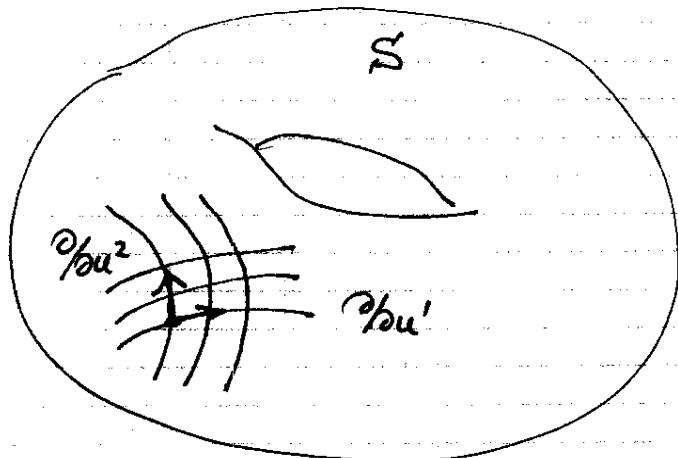
The integral has the form,

$$\int du^1 \dots du^s \underbrace{\omega_{\mu_1 \dots \mu_s}(x(u))}_{\omega} \frac{\partial x^{\mu_1}}{\partial u^1} \dots \frac{\partial x^{\mu_s}}{\partial u^s}.$$

The integrand (ω) can also be written,

$$\omega\left(\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^s}\right)$$

~~of~~ which is ω acting on the set of basis vectors on the submanifold S induced by the coordinate u^a .



In effect, these basis vectors $\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^s}$ span and define a small s -dimensional parallelopiped on the surface S . ω acts on this parallelopiped, producing a small contribution to the integral. The integral is the sum over the small parallelopipeds.

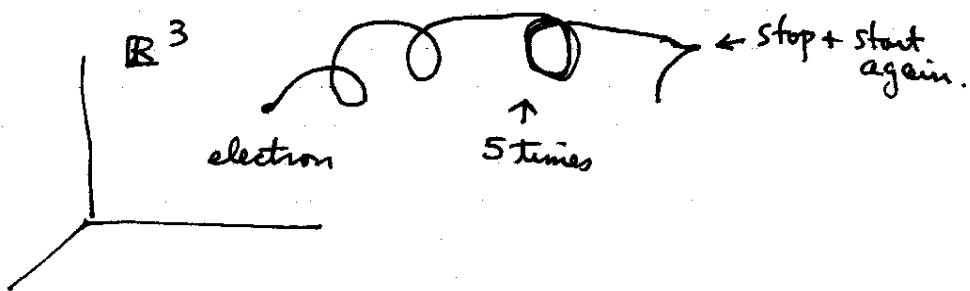
In Step 2 and Step 3, we require M or S to be orientable, because otherwise the integral depends on the coordinates used. If M or S are orientable, then the answer does not depend on the coordinates, apart from orientation.

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That is, two atlases of the same orientation give the same ~~sign~~ answer, one of the opposite orientation reverses the sign of the answer.

Step 4. Even in simple examples, it is easy to see that integrating over manifolds or submanifolds is not sufficient for real problems. Consider for example the work required to move an electron from one place to another in an electric field. This is a line integral (one-dimensional, using 1-forms). The path of the electron need not be a submanifold (one-dimensional). It may have self-intersections, the path may retrace or repeat itself, or the electron may stop for a while.



Obviously we want to parameterize the path by time or some other parameter, say, on the interval $I = [0, 1]$. Thus the path is a map,

$$f: I \rightarrow M \quad (= \mathbb{R}^3 \text{ for electron}).$$

and it is this map that we want to integrate over. The map f need not be injective and f_x need not have maximal rank.

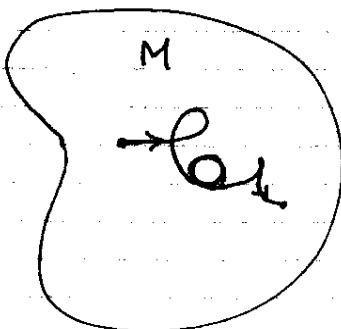
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We have seen that integrating r -forms over r -dimensional submanifolds is not general enough. For example, with $r=1$, we need to integrate over paths, that is, functions

$$f: I \rightarrow M$$

$$I = [0, 1] = \text{standard region } \subset \mathbb{R}$$



If $\omega = \omega_\mu(x) dx^\mu$ is a 1-form on M (in chart x^μ on M), then the integral we want is

$$\int_M \omega = \int_0^1 dt \quad \omega_\mu(x(t)) \frac{dx^\mu}{dt} = \int_I f^* \omega$$

where the last integral is that of a 1-form over a region of \mathbb{R}^r , defined previously (step 1 above). More generally, let us call a map

$$\sigma: I^r \rightarrow M$$

a singular r -cube. $I^r \subset \mathbb{R}^r$ is the r -cube, a standard region in \mathbb{R}^r ; the word "singular" is added to talk about the map σ , which need not be injective, nor need σ^* & σ_* have maximal rank. For example, $\text{im } \sigma$ (a subset of M) need not have dimension r , it may have self-intersections, etc. It need not be an r -dimensional submanifold of M .

Some authors (e.g. Nakahara) prefer to use a different standard region in \mathbb{R}^r , such as a simplex. Then the map is referred to as a singular simplex. There is no loss of generality in using cubes.

We now define the integral of an r -form $\omega \in \Omega^r(M)$ over a singular r -cube σ . It is $\int_{\sigma} \omega$. Here $\sigma: I^r \rightarrow M$.

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$$\int_{\sigma} \omega = \int_{I^r} \# \circ \sigma^* \omega$$

which reduces the integral to the integral of an r -form over an r -dimensional region of \mathbb{R}^r . To put this in coordinates, let x^μ be coordinates on M ($\mu = 1, \dots, m = \dim M$, $m \geq r$), and let u^d , $d = 1, \dots, r$ be the standard (Euclidean) coordinates on \mathbb{R}^r . Then

$$\int_{\sigma} \omega = \int_0^1 du^1 \dots \int_0^1 du^r \omega_{\mu_1 \dots \mu_r}(x(u)) \frac{\partial x^{\mu_1}}{\partial u^1} \dots \frac{\partial x^{\mu_r}}{\partial u^r}$$

The most general integral is taken over linear combinations of singular r -cubes. We consider only real coefficients here. If $\{\sigma_i^r\}$ is a set of singular r -cubes, then we define

$$c^r = \sum_i a_i \sigma_i^r, \quad a_i \in \mathbb{R}$$

as an r -chain. Integrals over r -chains are computed by

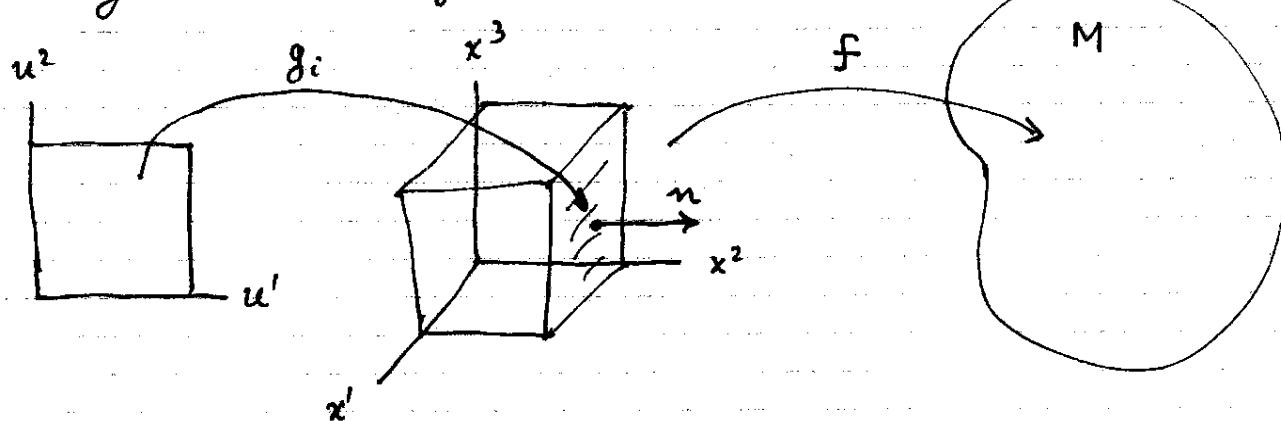
$$\int_{c^r} \omega = \sum_i a_i \int_{\sigma_i^r} \omega.$$

The set of all r -chains on M is the r -th chain group, $C_r(M, \mathbb{R})$ (we will drop the \mathbb{R} , it being henceforth understood.) The r -th chain group is a group \oplus in the sense that it is a vector space (an Abelian group). This is like the simplicial chain groups

considered earlier, except now they include singular cubes,
and now they are ∞ -dimensional.

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We now define the boundary operator, when acting on singular n -cubes. Once that is defined, ∂ becomes defined on chains by linearity. Consider e.g. $n=3$.



we have 6 faces, $i=1, \dots, 6$. Each face will be associated with a singular 2-cube. But a singular 2-cube is a map from the std 2-cube (square) in \mathbb{R}^2 to M , and the faces of $I^3 \subset \mathbb{R}^3$ are subsets of \mathbb{R}^3 , not \mathbb{R}^2 . So we introduce new maps $g_i: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ that map $I^2 \subset \mathbb{R}^2$ onto a face of $I^3 \subset \mathbb{R}^3$. Let x^μ be coords in \mathbb{R}^3 and u^μ be coords in \mathbb{R}^2 (or (x,y,z) and (u,v)). The map g_i must assign the right orientation to the face, defined by saying that $(n, \partial/\partial u^1, \partial/\partial u^2)$ are positively oriented in \mathbb{R}^3 , where n is an outward normal to the face, and $\partial/\partial u^1, \partial/\partial u^2$ span the face. For example, in the diagram above, we map I^2 onto the face of I^3 by writing,

$$\left. \begin{aligned} u &= z \\ v &= x \\ 1 &= y \end{aligned} \right\} .$$