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(3) $d^2 = 0$.

<u>Comments</u>. Property (2) is equivalent to defn of d (since it gives action of $d\alpha$ on arb. set of vectors). Turns out properties (1)+(3) also imply defn of d. Prop. (1) follows easily from "$d = \partial\wedge$" (you use a chain rule, but in order to bring the d in to act on β in the 2nd term, you must commute it through α, which introduces $(-1)^r$ factor.)

<u>Proof of property (3)</u>:

$$\alpha = \frac{1}{r!}\,\alpha_{\mu_1\ldots\mu_r}\; dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_r}$$

$$d\alpha = \frac{1}{r!}\,\alpha_{\mu_1\ldots\mu_r,\nu}\; dx^{\nu}\wedge dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_r} \qquad \text{(defn. of } d\text{)}.$$

$$dd\alpha = \frac{1}{r!}\,\alpha_{\mu_1\ldots\mu_r,\underbrace{\nu\sigma}_{\text{symm}}}\;\underbrace{dx^{\sigma}\wedge dx^{\nu}}_{\text{antisymm}}\wedge\ldots\wedge dx^{\mu_r} = 0.$$

<u>Special cases of (2)</u>:

$r=0$, $f \in \mathcal{F}(M)$.

$$df(X) = Xf.$$

$r=1$, $\alpha \in \Omega^1(M)$

$$d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])$$

$r=2$, $\beta \in \Omega^2(M)$

$$d\beta(X,Y,Z) = X\beta(Y,Z) - Y\beta(X,Z) + Z\beta(X,Y)$$
$$- \beta([X,Y],Z) + \beta([X,Z],Y) - \beta([Y,Z],X).$$

(4) Another property of d. Let $f: M \to N$ be a map (not nec. a diffeo.). Let $\alpha \in \Omega^r(N)$, so $f^*\alpha \in \Omega^r(M)$. Then 10/9/08

$$\boxed{f^*(d\alpha) = d(f^*\alpha)} \quad d \text{ commutes with pull-backs.}$$

Easy to prove in components/coordinates.

<u>important terminology.</u>

An r-form $\alpha \in \Omega^r(M)$ is <u>closed</u> if $d\alpha = 0$.

It is <u>exact</u> if $\exists\, \beta \in \Omega^{r-1}(M)$ such that $\alpha = d\beta$.

Now we consider the <u>interior product</u>. Let $X \in \mathfrak{X}(M)$, then the interior product is an operator $i_X : \Omega^r(M) \to \Omega^{r-1}(M)$, defined by ($\alpha \in \Omega^r(M)$):

$$(i_X\alpha)(Y_1, \dots, Y_{r-1}) = \alpha(X, Y_1, \dots, Y_{r-1}).$$

This is a purely algebraic operation (just insert X into 1st slot of α), no differentiation required. Notice that i_X lowers the rank of α, while d raises it. Properties of i_X:

(1) $i_X(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta + (-1)^r \alpha \wedge (i_X\beta)$, $\quad \alpha \in \Omega^r(M)$

(2) $i_X^2 = 0$

(3) $\boxed{\mathcal{L}_X = i_X d + d i_X}$ (Acting on forms).

Property (3) is the Cartan formula. The geometrical meaning of this formula must wait until we cover Stokes' theorem. A proof is given in the book. The proof I prefer runs along

these lines:

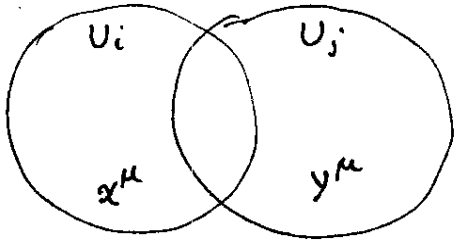
① Show that the Cartan formula works for $r=0$ and $r=1$.

② Show that $i_X d + d i_X$ is a <u>derivation</u> (obeys Leibnitz) when acting on $\wedge$ products.

These steps are straightforward. They prove the Cartan formula because an arbitrary form can be represented as a linear combination of $\wedge$ products of 1-forms and 0-forms.

Now an introduction to the integration of differential forms. General idea is that r-forms get integrated over r-dimensional submanifolds of M. (Actually, the objects that they get integrated over are more general than submanifolds, they are <u>chains</u>. More about that later.) For now we concentrate on a special case, integrating m-forms on an m-dimensional manifold.

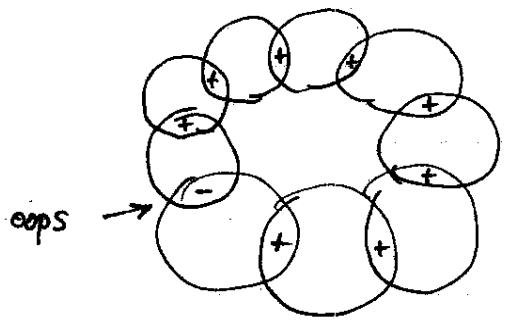
First, integration over a manifold is not meaningful unless the manifold is orientable. Consider 2 charts that overlap. ($m = \dim M$).



$\varphi_i : U_i \to \mathbb{R}^m$

In the overlap region, the Jacobian $\frac{\partial x^\mu}{\partial y^\nu}$ is nonsingular, so $\det\left(\frac{\partial x^\mu}{\partial y^\nu}\right)$ is either >0 or <0. If it is >0, then we say the two charts have the same orientation. Is it possible to cover M with charts that have the same orientation? Depends on M. Those manifolds that can be covered with charts

that have the same orientation are said to be <u>orientable</u>. For some manifolds, however, this cannot be done.



Example of Möbius strip, $\mathbb{RP}^2$

There is a relation between orientability and m-forms on M. Let $\phi \in \Omega^m(M)$. Then

$$\phi = \frac{1}{m!}\phi_{\mu_1 \dots \mu_m}(x)\, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_m}$$

$$= \underbrace{\phi_{12\dots m}(x)}\, dx^1 \wedge \dots \wedge dx^m$$

↳ Call this $\rho(x)$, it is the one (indep) component of ϕ. w.r.t. chart x^μ.

change charts, $x^\mu \to y^\mu$. Then

$$dx^\mu = \frac{\partial x^\mu}{\partial y^\nu}\, dy^\nu.$$

$$dx^1 \wedge \dots \wedge dx^m = \det\left(\frac{\partial x}{\partial y}\right) dy^1 \wedge \dots \wedge dy^m,$$ because of antisymmetry.

So, under the change of coordinates $x^\mu \to y^\mu$, we find $\rho \longrightarrow \rho \det\left(\frac{\partial x}{\partial y}\right)$. We say that ρ transforms as a <u>pseudo-scalar</u>. The value of ρ is not preserved under a change of coordinates, but if x^μ and y^μ have the same orientation, then the <u>sign</u> of ρ is conserved

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Related to a thm: An m-form ϕ exists on M that is nonzero everywhere iff M is orientable. This is easily proved using partitions of unity, discussed in book. Note that ϕ vanishes at a point $x \in M$ iff $\rho(x) = 0$.

If M is orientable, then we can construct atlases in which all the charts have the same orientation. Call these "oriented atlases". If we have two atlases, their charts are all either oriented the same or oriented oppositely (if they overlap). Thus the space of oriented atlases consists of two equivalence classes. We may call one of these "positively oriented" and the other "negatively oriented", but this is just a convention, not an absolute designation.

In order to specify an orientation, we choose an oriented atlas. Actually it suffices to choose a single chart, or even just a basis $\{e_\mu\}$ in a single tangent space at a single point, since if M is orientable this will ~~back~~ fix the orientations of all other charts.

Let $\omega \in \Omega^m(M)$, and suppose we have an oriented atlas chosen.

~~Then use triangulation so that each~~ Then to integrate ω over a region contained in one chart x^μ we define

$$\int_R \omega = \int dx^1 \ldots dx^m$$

Let $\phi : U \to V \subset \mathbb{R}^m$ be a chart containing a region R ($R \subset U$) over which we wish to integrate ω.

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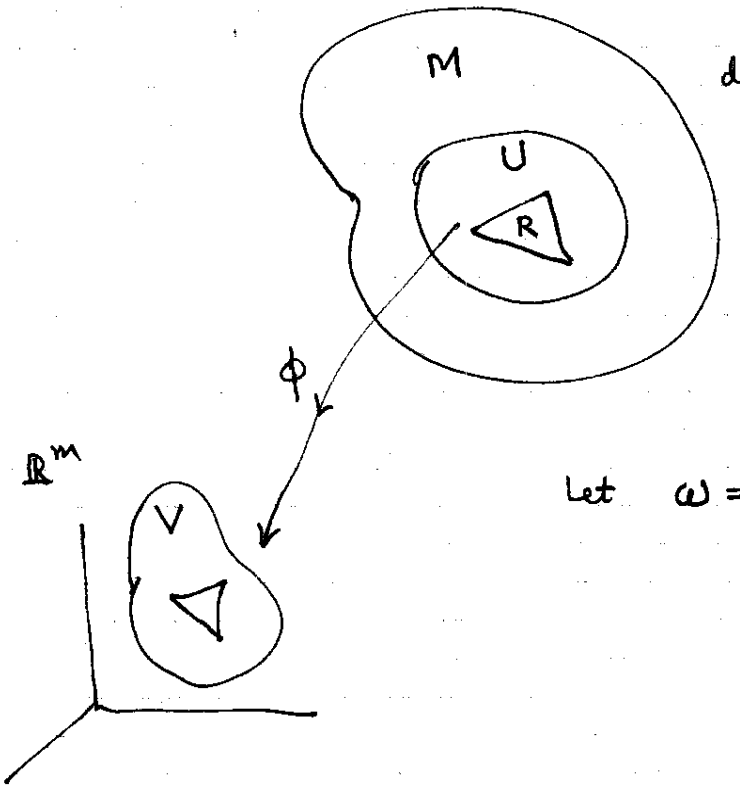

$\dim M = m$

$V = \phi(U)$

Let $\omega = \rho(x)\, dx^1 \wedge \ldots \wedge dx^m$

Then we define

$$\int_R \omega = \int_{\phi(R)} \rho(x)\; dx^1 \ldots dx^m .$$

The final integral is a normal Riemann integral (in particular, the answer does not depend on the ordering of the dx's.) By breaking M up into regions such that each region lies in one chart, we can add the integrals up to get $\int_M \omega$. The answer is independent of the oriented atlas we choose, apart from sign. Note that the answer is independent of the coordinates you choose, as long as the orientation is the same.

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Some applications of differential forms to classical mechanics.

Let P = the phase space of a mechanical system, with coordinates (q_i, p_i), $i=1,\dots,n$, $\dim P = 2n$, where n = number of degrees of freedom.

Aside: Often $P = T^*Q$, where Q = the configuration space. It takes some thought to understand this, and we won't need it for the following developments. Let's just imagine that P is some space with coordinates (q_i, p_i).

The Hamiltonian is a scalar field on P, $H: P \to \mathbb{R}$, and Hamilton's equations in coordinates q_i, p_i are

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned}\right\}.$$

Obviously this creates a map from scalar fields (H) to vector fields $(\dot{q}, \dot{p})$ on P:

$$\text{map: } \mathcal{F}(P) \to \mathfrak{X}(P) : H \to X_H,$$

where X_H is the vector field which in coords q_i, p_i has components

$$X_H = \begin{pmatrix} \dot{q}_i \\ \dot{p}_i \end{pmatrix} = \begin{pmatrix} \partial H/\partial p_i \\ -\partial H/\partial q_i \end{pmatrix}.$$

Consider 1-forms on P. If $\alpha \in \Omega^1(P)$, then α can be written,

$$\alpha = \sum_i \left[a_i(q,p)\, dq_i + b_i(q,p)\, dp_i \right]$$

where a_i, b_i are the components of α w.r.t. coords q_i, p_i.

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Here is a special 1-form, $\theta \in \Omega^1(P)$,

$$\theta = \sum_i p_i \, dq_i ,$$

special because $b_i = 0$ and $a_i(q,p) = p_i$. We just pull θ out of the air, but it can be justified as a natural 1-form on P in the case $P = T^*Q$. Also define

$$\omega = d\theta = \sum_i dp_i \wedge dq_i \quad , \qquad \omega \in \Omega^2(P).$$

Hamilton's equations connect $H \in \mathcal{F}(P)$ with $X_H \in \mathfrak{X}(P)$. The connection can written in the form (it turns out),

$$\boxed{i_{X_H}\omega + dH = 0}$$

which is a modern, coordinate-free way of writing Hamilton's equations. Notice that ω is a 2-form so $i_{X_H}\omega$ is a 1-form.

We'll show the equivalence of this (modern version) to the traditional version of Hamilton's equations by working in coordinates.

Write $z^\mu = \begin{pmatrix} q_i \\ p_i \end{pmatrix}$ for a collective notation for coordinates on P, where $\mu = 1, \ldots, 2n$. Write

$$\omega = \frac{1}{2}\omega_{\mu\nu}\, dz^\mu \wedge dz^\nu \qquad \text{(summation convention on Greek indices)}$$

for the 2-form ω, where $\omega_{\mu\nu} = \omega\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right)$ is the component matrix. This is a $2n \times 2n$ matrix that we can partition into four $n \times n$ blocks,

$$\omega_{\mu\nu} = \begin{array}{c c} & \begin{array}{cc} q & p \end{array} \\ \begin{array}{c} q \\ p \end{array} & \left(\begin{array}{c|c} & \\ \hline & \end{array}\right) \end{array} .$$

To get the q-p block we compute,

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$$\omega_{q_i p_j} = \omega\left(\frac{\partial}{\partial q_i}, \frac{\partial}{\partial p_j}\right) = \sum_k (dp_k \wedge dq_k)\left(\frac{\partial}{\partial q_i}, \frac{\partial}{\partial p_j}\right)$$

$$= \sum_k \left[dp_k\left(\frac{\partial}{\partial q_i}\right) dq_k\left(\frac{\partial}{\partial p_j}\right) - dp_k\left(\frac{\partial}{\partial p_j}\right) dq_k\left(\frac{\partial}{\partial q_i}\right)\right]$$

$$= \sum_k \left[0 \cdot 0 - \delta_{kj}\,\delta_{ik}\right] = -\delta_{ij} \qquad (-I \text{ matrix})$$

The p-q block is $+I$ matrix by anti-symmetry, and the qq and pp blocks vanish. So,

$$\omega_{\mu\nu} = \left(\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array}\right).$$

Now check $i_{X_H}\omega + dH = 0$. Put it in coordinates,

$$X_H^\mu\, \omega_{\mu\nu} + H_{,\nu} = 0,$$

or, since $\omega_{\mu\nu} = -\omega_{\nu\mu}$,

$$\omega_{\mu\nu}\, X_H^\nu = H_{,\mu}.$$

Now check it:

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} \dot q \\ \dot p \end{pmatrix} \overset{?}{=} \begin{pmatrix} \partial H/\partial q \\ \partial H/\partial p \end{pmatrix} = \begin{pmatrix} -\dot p \\ +\dot q \end{pmatrix},$$

so it works. Here we use

$$X_H^\mu = \begin{pmatrix} \dot q_i \\ \dot p_i \end{pmatrix} = \begin{pmatrix} \partial H/\partial p_i \\ -\partial H/\partial q_i \end{pmatrix}.$$

Usually we are given H and want to find the eqns. of motion, i.e., X_H. But $i_{X_H}\omega + dH = 0$ only gives X_H implicitly. To explicitly solve for X_H, look at the coordinate version of this eqn, $\omega_{\mu\nu} X_H^\nu = (dH)_\mu = H_{,\mu}$. Obviously we must invert the matrix $\omega_{\mu\nu}$. This is possible, since $\det \omega_{\mu\nu} = +1$ in the q, p coordinates. Define $J^{\mu\nu}$ by

$$J^{\mu\nu}\omega_{\nu\alpha} = \delta^\mu_\alpha ,$$

so that $J^{\mu\nu}$ is the inverse matrix of $\omega_{\mu\nu}$. It is also the components of an antisymmetric, rank 2, <u>contravariant</u> tensor,

call it this.

$$J = J^{\mu\nu} \frac{\partial}{\partial z^\mu} \otimes \frac{\partial}{\partial z^\nu} = \underline{\text{Poisson}}\ \underline{\text{tensor}}.$$

It can be thought of as a map,

$$J: \mathfrak{X}^*(P) \times \mathfrak{X}^*(P) \to \mathcal{F}(P).$$

Then solving Hamilton's eqns for X_H, we get

$$X_H^\mu = \dot{z}^\mu = J^{\mu\nu} H_{,\nu} .$$

Now a definition. A manifold M plus $\omega \in \Omega^2(M)$ such that

① $d\omega = 0$ (ω is closed)

② $\det \omega_{\mu\nu} \neq 0$ (ω is "nondegenerate")

is a <u>symplectic</u> <u>manifold</u>. ω is called the <u>symplectic</u> 2-form.

We see that P is a symplectic manifold. This holds whenever $P = T^*Q$, but there are also symplectic manifolds that are not cotangent bundles. In fact, Lie groups are associated with families of symplectic manifolds that are connected with the representation theory of the groups. On any symplectic manifold, the Poisson tensor $J^{\mu\nu}$ exists.

The condition (2) is stated w.r.t. a coordinate system ($\det \omega_{\mu\nu} \neq 0$) but is independent of coordinates. A coordinate-free way of writing it is to say $\omega|_z (X, Y) = 0$ for all Y iff $X = 0$, for all $z \in P$. Here $X, Y \in T_z P$.

Let us consider the rate of change of $f: P \to \mathbb{R}$ (a "classical observable") along the flow X_H. It is

$$X_H f = X_H^\mu f_{,\mu} = J^{\mu\nu} H_{,\nu} f_{,\mu} = \{f, H\},$$

where $\{\ ,\ \}$ is the Poisson bracket:

$$\{A, B\} = A_{,\mu} J^{\mu\nu} B_{,\nu} = \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right)$$

Note that in q-p coordinates, the component matrix $J^{\mu\nu}$ is

$$J^{\mu\nu} = \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array} \right).$$

We can also write,

$$\{A, B\} = J(dA, dB), \quad \text{thinking of}$$

$$J: \mathfrak{X}^*(P) \times \mathfrak{X}^*(P) \to \mathcal{F}(P).$$

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The Poisson bracket is usually thought of as a map,

$$\{\ ,\ \} : \mathcal{F}(P) \times \mathcal{F}(P) \to \mathcal{F}(P).$$

It has the following properties:

① $\{A, B\} = -\{B, A\}$

② $\{A, B\}$ is linear in both operands (over $\mathbb{R}$)

③ $\{\{A, B\}, C\} + \circlearrowleft = 0$ (Jacobi).

Thus, the P.B. endows the set $\mathcal{F}(P)$ with the structure of a <u>Lie algebra</u>. There is the question as to what is the corresponding group.

The Jacobi identity holds on any symplectic manifold. It is an exercise to prove it.

As explained, Hamilton's equations provide a map $: \mathcal{F}(P) \to \mathfrak{X}(P) : A \to X_A$, where now we write A instead of H because the function need not be the Hamiltonian in the usual sense. A is called the <u>Hamiltonian function</u> and X_A the <u>Hamiltonian vector field</u>. The set of all Hamiltonian vector fields is a subset of $\mathfrak{X}(P)$, the set of all vector fields on P.

When we write $i_{X_A}\omega + dA = 0$ in coordinates,

$$\omega_{\mu\nu} X_A^{\nu} = A_{,\mu}$$

we see that $\omega_{\mu\nu}$ is being used just as the covariant metric tensor $g_{\mu\nu}$ is used on a metric space to "lower an index."

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That is, we can think of ω at $z \in P$ as a map

$$\omega|_z : T_z P \times T_z P \to \mathbb{R},$$

or a closely associated map,

$$\omega|_z : T_z P \to T_z^* P : X^\mu \mapsto \omega_{\mu\nu} X^\nu,$$

$$\text{i.e. } X \mapsto -i_X \omega.$$

Similarly, in Hamilton's eqns in the form

$$\dot{z}^\mu = X_A^\mu = J^{\mu\nu} A_{,\nu},$$

$J^{\mu\nu}$ is being used to "raise an index", just like the contravariant metric tensor $g^{\mu\nu}$ on a Riemannian manifold.

The set of all vector fields on P, $\mathfrak{X}(P)$, forms a Lie algebra under the Lie bracket, $[\ ,\ ]$. What about the subset of Hamiltonian vector fields? They form a Lie algebra by themselves under the Lie bracket, as shown by the formula,

$$[X_A, X_B] = -X_{\{A,B\}}.$$

It is an exercise to prove this formula. This formula displays a Lie algebra anti-homomorphism between $\mathcal{F}(P)$ and the set of Hamiltonian vector fields on P ("anti" refers to the $-$ sign).

Denote the set of Hamiltonian vector fields on P by $\mathfrak{X}_H(P)$. As explained, this is a Lie algebra under $[\ ,\ ]$. What is the corresponding group? It is the group generated by

Hamiltonian advance maps.

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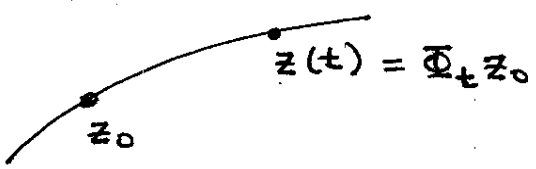
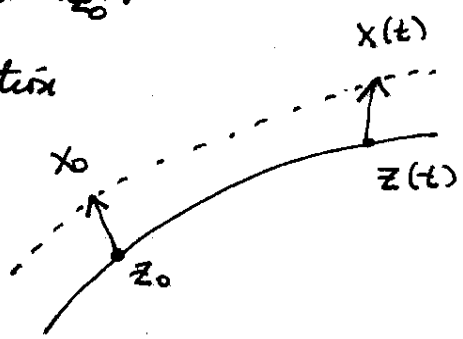
Here is a problem where those advance maps occur. Suppose we have an orbit of a mechanical system in phase space (an integral curve of X_H) with initial conditions z_0. Let Φ_t be the advance map.



Suppose we wish to make a small perturbation in initial conditions, for example, a mid-course correction in the trajectory of a space probe. Denote the small change at z_0 by X_0, which we can think of as a vector in $T_{z_0}P$:

We want to find the perturbation $X(t)$ in the final conditions which result. We



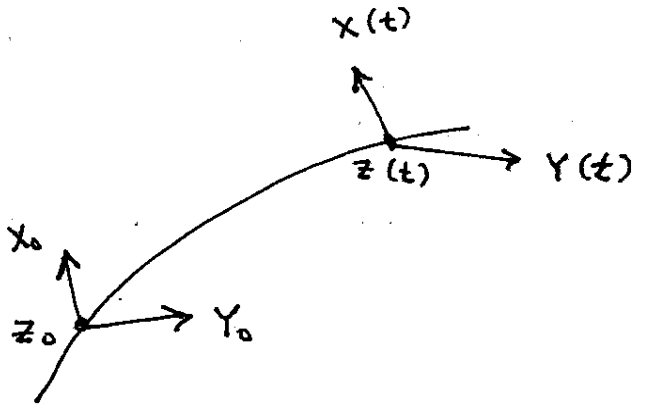
must map both the base and tip of X_0 under Φ_t to get $X(t)$. This means

$$X(t) = \Phi_{t*} X_0 ,$$

i.e. we use the tangent map of the advance map. Now suppose we have two small changes in initial conditions and corresponding changes in the final conditions.

so

$$X(t) = \Phi_{t*} X_0$$
$$Y(t) = \Phi_{t*} Y_0.$$



Now consider the action of the symplectic form ω, eval. at $z(t)$, on $X(t)$ and $Y(t)$:

$$\omega|_{z(t)}\left(X(t), Y(t)\right) = \omega|_{z(t)}\left(\Phi_{t*} X_0, \Phi_{t*} Y_0\right)$$

$$= \left(\Phi_t^* \omega\right)\Big|_{z_0} \left(X_0, Y_0\right)$$

by the definition of the pull-back.

Now it turns out that

$$\Phi_t^* \omega = \omega,$$

the symplectic form is invariant under Hamiltonian advance maps. This means that

$$\omega|_{z(t)}\left(X(t), Y(t)\right) = \omega|_{z_0}\left(X_0, Y_0\right).$$

The "symplectic area" spanned by vectors X, Y is independent of time. This is a basic property of nearby orbits in classical mechanics.

To prove it, consider $\frac{d}{dt} \Phi_t^* \omega\Big|_{t=0} = \mathcal{L}_X \omega,$

which is the definition of $\mathcal{L}_X$. By the Cartan formula, 10/9/08

$$\mathcal{L}_X \omega = i_X d\omega + d i_X \omega,$$

where $X = X_H$. But $d\omega = 0$ (ω is symplectic), and $i_X\omega = -dH$ so $d i_X \omega = -ddH = 0$. So $\mathcal{L}_X \omega = 0$, and

$$\frac{d}{dt} \Phi_t^* \omega \Big|_{t=0} = 0.$$

This holds at z_0. But since z_0 is arbitrary, it holds everywhere along an orbit, so $\Phi_t^* \omega$ is indep. of t, hence equal to its value at $t=0$, $\Phi_t^* \omega = \omega$.

Not only is ω preserved along Hamiltonian flows, but so is any power of ω. For example,

$$\mathcal{L}_X (\omega \wedge \omega) = \mathcal{L}_X \omega \wedge \omega + \omega \wedge \mathcal{L}_X \omega \quad \text{(Leibnitz)}$$
$$= 0.$$

So all the forms, ω, $\omega \wedge \omega, \ldots,$ $\omega \wedge \ldots \wedge \omega$ (n times) are preserved along flows. The last form,

$$\omega \wedge \ldots \wedge \omega \ (n \text{ times}) = \omega^n$$

is a $2n$-form on $2n$-dimensional P. In $q\,p$ coordinates, it is

$$\omega^n = (\text{const})\ dq_1 \wedge \ldots \wedge dq_n \wedge dp_1 \wedge \ldots \wedge dp_n$$

i.e., it is the volume form on phase space. The fact that it is preserved under Hamiltonian flows is sometimes called <u>Liouville's theorem</u>.