Recall, a tensor of type \((r,s)\) at a point \(p \in M\) is a multilinear map,

\[
T: \underbrace{T^*_p M \times \ldots \times T^*_p M}_r \times \underbrace{T_p M \times \ldots \times T_p M}_s \to \mathbb{R}.
\]

The components of \(T\) are given by \((\text{w.r.t. a chart})\)

\[
T^{i_1 \ldots i_r}_{j_1 \ldots j_s} = T(\partial x^{i_1}, \ldots, \partial x^{i_r}; \frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_s}}).
\]

A tensor field on \(M\) is an assignment of a tensor at each point \(p \in M\). The components of a tensor field are functions of position.

Now we consider the behavior of fields under maps. Let \(f: M \to N\) be a map between manifolds, let \(p \in M\) and \(q \in N\) be points such that \(q = f(p)\).

\[\begin{array}{c}
\text{M} \\
\cdot p \\
\end{array}
\text{N} \xleftarrow{\text{dim } M = m} \xrightarrow{\text{dim } N = n} \text{f}
\]

\[\begin{array}{c}
\cdot q \\
\end{array}
\]

**Question:** Given \(X \in T_p M\), is there any way to associate it with a \(Y \in T_q N\)? Yes, use the small displacement interpretation of a tangent vector \((p, \text{close to } p)\), and define \(q_1 = f(p_1)\).

(You map both the base and the tip of the small arrow under \(f\),...
If we impose coordinate (charts) on $M$ and $N$, we can write $f^*$ in coordinates. Let $q^i$ be coordinates on $M$ and $g^i$ on $N$. Then we can say, if $X = \mathcal{C}$, then $Y = f_\mathcal{C}$. Thus define 

\[ f_* : T_N \to T_M \]

Thus we can say, if $X = \mathcal{C}$, then $Y = f_\mathcal{C}$. Thus define

\[ f_* : T_N \to T_M \]

Alternatively, in the equivalence class of curves interpretation, just map the curves themselves.

to get a new small arrow on $N$. Both are understood to be displacements taking place in some aligned parameter $t$. Just get there on $M$. Let $g^i$.

\[ \frac{dx^i}{dt} = \frac{dy^i}{dt} \]

\[ x = \mathcal{C}, \quad y = \mathcal{C} \]

\[ f_* : [C] \to [f_\mathcal{C}] \]

\[ f_* : T_N \to T_M \]

\[ \mathcal{C} \]
Then
\[ Y^i = \sum_j \frac{\partial Y^i}{\partial x^j} X^j, \]

where \( \frac{\partial Y^i}{\partial x^j} \) is the coordinate representation of the derivatives of \( f \).

This is how we map vectors. For covectors, it works the other way, i.e., given a covector \( \alpha \in T_q^* N \), we can associate it with another covector \( \beta \in T_p^* M \).

That is, we define a map \( f^*: T_q^* N \to T_p^* M \) by demanding
\[ \beta(X) = \alpha(Y) \text{ when } Y = f_* X. \]
That is, \( \beta = f^* \alpha \) is defined by
\[ (f^* \alpha)(X) = \alpha(f_* X), \quad \forall X \in T_p M. \]

The covector \( f^* \alpha \in T_p^* M \) is said to be the pull-back of \( \alpha \in T_q^* N \), because it works in the opposite direction to \( f \).

This is for a covector at a point (two of them, \( \alpha \) and \( f^* \alpha \)).

If now we let \( \alpha \) be a covector field (same symbol, new meaning), \( \alpha \in \mathfrak{X}^*(N) \), then \( f^* \alpha \in \mathfrak{X}^* M \), is given by