Recall, a tensor of type (r,s) at a point $p \in M$ is a multilinear map,

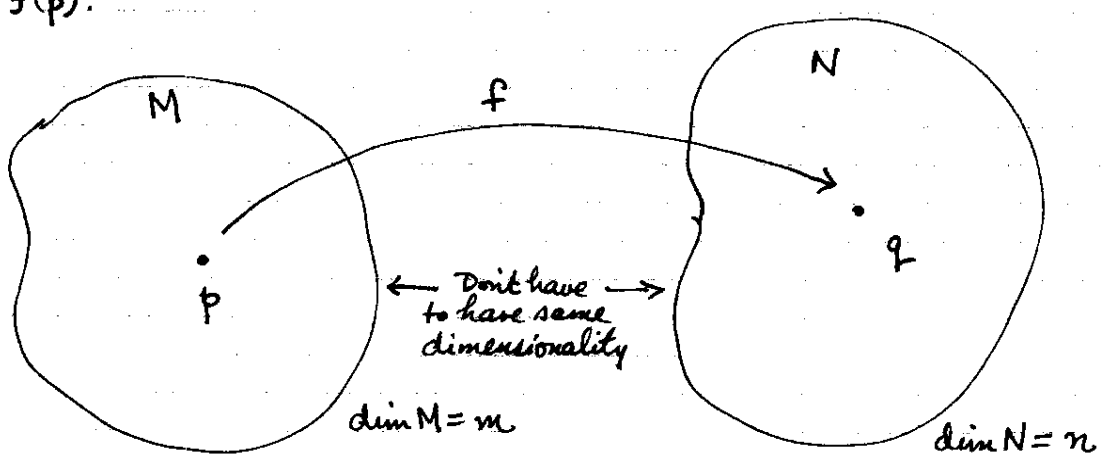
$$T: \underbrace{T_p^* M \times \dots \times T_p^* M}_{r \text{ copies}} \times \underbrace{T_p M \times \dots \times T_p M}_{s \text{ copies}} \to \mathbb{R}.$$

The <u>components</u> of T are given by (w.r.t. a chart)

$$T^{i_1 \dots i_r}{}_{j_1 \dots j_s} = T\left(dx^{i_1}, \dots, dx^{i_r}; \frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_s}}\right).$$

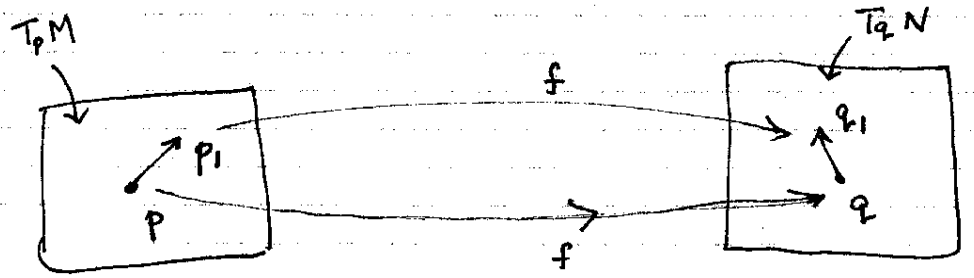
A <u>tensor field</u> on M is an assignment of a tensor at each point $p \in M$. The components of a tensor field are functions of position.

Now we consider the <u>behavior of fields under maps</u>. Let $f: M \to N$ be a map between manifolds, let $p \in M$ and $q \in N$ be points such that $q = f(p)$.

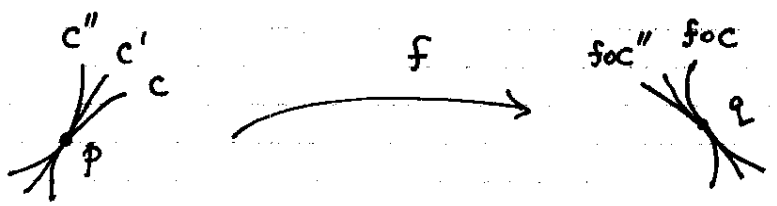


<u>Question</u>: Given $X \in T_p M$, is there any way to associate it with a $Y \in T_q N$? Yes, use the small displacement interpretation of a tangent vector (p_1 close to p), and define $q_1 = f(p_1)$. (You map both the base and the tip of the small arrow under f,

to get a new small arrow on N. Both are understood to be displacements taking place in some elapsed parameter Δt.)



Alternatively, in the equivalence class of curves interpretation, just map the curves themselves:



Thus we can say, if $X = [c]$, then $Y = [f \circ c]$. This defines a linear map

$$f_* : T_pM \to T_qN$$

where f_* is called the <u>tangent map</u>, <u>derivative map</u>, or <u>push-forward</u>. Note that f_* can be defined succinctly by

$$f_*[c] = [f \circ c].$$

If we impose <u>coordinates</u> (charts) on M and N (containing p and q), we can write f_* in coordinates. Let x^i be coordinates on M and y^i those on M. Let

$$X = \sum_{i=1}^{m} X^i \left.\frac{\partial}{\partial x_i}\right|_p$$

$$f_*X = \quad Y = \sum_{i=1}^{n} Y^i \left.\frac{\partial}{\partial y^i}\right|_q$$

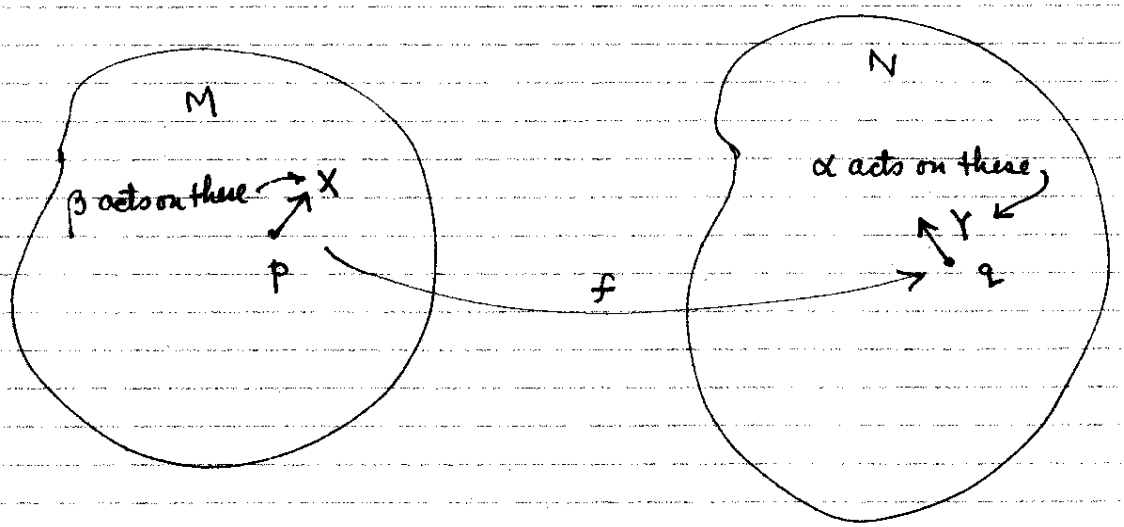
Then

this is just the chain rule, if you think $X^i = \frac{dx^i}{dt}$, $Y^i = \frac{dy^i}{dt}$.

$$Y^i = \sum_j \frac{\partial y^i}{\partial x^j} X^j ,$$

where $\partial y^i/\partial x^j$ is the coordinate representation of the derivatives of f.

This is how we map vectors. For covectors, it works the other way, i.e., given a covector $\alpha \in T_q^* N$, we can associate it with another covector $\beta \in T_p^* M$.



That is, we define a map $f^*: T_q^* N \to T_p^* M$ by demanding $\beta(X) = \alpha(Y)$ when $Y = f_* X$. That is, $\beta = f^*\alpha$ is defined by

$$(f^*\alpha)(X) = \alpha(f_* X), \qquad \forall\ X \in T_p M.$$

The covector $f^*\alpha \in T_p^* M$ is said to be the <u>pull-back</u> of $\alpha \in T_q^* N$, because it works in the opposite direction to f.

This is for a covector at a point (two of them, α and $f^*\alpha$). If now we let α be a covector field (same symbol, new meaning), $\alpha \in \mathcal{X}^*(N)$, then $f^*\alpha \in \mathcal{X}^* M$, is given by