Now we examine the higher homotopy groups.
Recall that to study maps: \( S^1 \to M \), we studied maps: \( I \to M \) (\( I = [0, 1] \subset \mathbb{R} \)), subject to the condition that \( f(0) = f(1) \) (these are loops). This was a matter of convenience.

\[
I = \begin{array}{c}
\text{body of } I \text{ mapped to same pt.}
\end{array}
\]

Similarly, to study maps: \( S^2 \to M \) it is convenient instead to look at maps

\[
a: I \times I \to M,
\]

where \( I \times I \) is the square and it is understood that the boundary of \( I \times I \) is mapped to a single pt.

Note, square with all pts on bdy \( I \times I \) identified \( \cong S^2 \). Call such a map a 2-loop. Then the rest of the story proceeds very much as in the case of \( \pi_1(M) \):

More generally, consider maps

\[
a: I \times \ldots \times I \to M
\]

(these are \( n \)-loops).

1. \( a \sim \beta \), 2-loops \( a \) and \( \beta \) are homotopic, if \( \exists \) a smooth interpolating map (the homotopy) that preserves the boundary point. (i.e., maps the bdy to \( x_0 \) for all values of the deform param.)

2. \( a \sim \beta \) is an equivalence relation, hence classes \([a], [\beta]\) etc. meaningful.

3. \( a \times \beta \) is defined by

\[
(a \times \beta)(s_1, s_2) = \begin{cases} 
\alpha(2s_1 \cdot s_2) & 0 \leq s_1 \leq 1,
\beta(2s_1 - 1, s_2) & \frac{1}{2} \leq s_1 \leq 1.
\end{cases}
\]
Thus, pictorially:

4. \([\alpha]*[\beta]\) is defined (* respects homotopy classes), and other axioms of a group are satisfied. The group (for \(n\)-loops) is denoted \(\pi_n(M,x_0)\).

5. \(\pi_n(M,x_0)\) is isomorphic to \(\pi_n(M,x)\), if \(M\) is arcwise connected. Hence we just write \(\pi_n(M)\) for the abstract group (the \(n\)-th homotopy group).

6. If \(X\) is of same homotopy type as \(Y\), then \(\pi_n(X) = \pi_n(Y)\).

7. \(\pi_n(X \times Y) = \pi_n(X) \times \pi_n(Y)\).

There is one important property of the higher homotopy groups (\(n \geq 2\)) not shared by \(\pi_1\): The higher homotopy groups are Abelian. Reason for this can be seen pictorially:

where the shaded region (and all boundaries) are mapped into \(x_0\).
This shows that $\alpha \ast \beta = \beta \ast \alpha$, hence $[\alpha] \ast [\beta] = [\beta] \ast [\alpha]$.

Another property of higher homotopy groups not shared by $\pi_1$ concerns the universal covering space. Let $M$ be a space and $\tilde{M}$ its universal cover. Then $\pi_1(\tilde{M}) = \{ e \}$ (if $M$ is simply connected), but in general $\pi_1(M)$ is not trivial (if it is trivial, then $M = \tilde{M}$).

But for $n \geq 2$, $\pi_n(\tilde{M}) = \pi_n(M)$. The basic idea behind this fact is that the spheres $S^n$ are simply connected for $n \geq 2$, so it's possible to define the lift of an $n$-loop:

The lift of a point $x$ on the $2$-loop $\alpha$ is the equivalence class $[(x, y)]$, where $y$ is a path from $x_0$ to $x$ confined to $\alpha$. Since the $2$-loop $\alpha$ is simply connected, it means that the class $[(x, y)]$ is independent of the choice of $y$, and therefore specifies a unique path in $\tilde{M}$.

As an application of this, note that $RP^n$ is covered by $S^n$, so

$$\pi_k(RP^n) = \pi_k(S^n).$$

So what is $\pi_k(S^n)$?
Simplest case is $\pi_n(S^n)$, which concerns mappings of $S^n \to S^n$. Recall $\pi_1(S^1) = \mathbb{Z}$, where $n \in \mathbb{Z}$ is interpreted as a winding number. There is a generalization of this to higher dimensions, i.e., for a map $f : S^n \to S^n$ it is possible to say "how many times" the image of $f$ "wraps around" $S^n$. This number is called the Brouwer degree of degree of $f$. And, as in the case $n=1$, it turns out that the Brouwer degree uniquely characterizes the homotopy classes. Thus,

$$\pi_n(S^n) = \mathbb{Z} \quad (\text{all } n \geq 1).$$

What about the case $\pi_k(S^n)$ for $k < n$? Recall $\pi_1(S^n) = \{e\}$ for $n \geq 2$ (the loop is contractible on the face of $S^n$). Something like this also happens for $\pi_k(S^n)$ when $k < n$, that is,

$$\pi_k(S^n) = \{e\}, \quad 1 \leq k \leq n.$$  

It turns out the case $k > n$ is also interesting. It's not easy to see this, because the highest dimensional case that is easy to visualize concerns maps $f : S^2 \to S^1$ ($k=2, n=1$), and all such maps are trivial (= contractible). (Think of mapping a sphere $S^2$ onto the equator.) Thus,

$$\pi_2(S^1) = \{e\}.$$  

But it turns out there are nontrivial (not homotopic to constant map) maps from $S^3$ to $S^2$. The Hopf map discussed earlier in the
course is an example of one of these. In fact,

\[ \pi_3(S^2) = \mathbb{Z} \]

In all these results, \( k>1 \), we can replace \( S^n \) by \( \mathbb{R}P^n \).

see the table in Nakahara, p.151.

Finally, what about the case \( n=0 \)? A literal interpretation of the definition of \( \pi_k(M; \mathbb{C}) \) to the case \( n=0 \) would involve maps \( f : S^0 \to M \) with one point fixed. \( S^0 \) is properly just 2 points (\( \pm 1 \) in \( \mathbb{C} \)), so we can say \( f(1) = x_0, f(-1) = \) another pt. \( x_1 \), say.
But if \( M \) is connected, then \( x_1 \) can be continuously pulled to \( x_0 \), and \( \pi_0(M) = f \pi^3 \), all connected \( M \).

Now go back to defects in CM systems, now that we know about higher homotopy groups. Make a table of various order parameter spaces (OPS's):

<table>
<thead>
<tr>
<th>System</th>
<th>OPS = M</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liqui. 4He</td>
<td>( S^1 )</td>
</tr>
<tr>
<td>xy spin model</td>
<td></td>
</tr>
<tr>
<td>Nematic liquids</td>
<td>( \mathbb{R}P^2 )</td>
</tr>
<tr>
<td>Dipole locked ( ^3He )</td>
<td>( SO(3) = \mathbb{R}P^3 )</td>
</tr>
<tr>
<td>phase A</td>
<td></td>
</tr>
</tbody>
</table>

Recall, point defects are described by non-trivial homotopy classes \( \pi_2(M) \), line defects (vortices) by \( \pi_1(M) \), and textures (field configurations) with asymptotically constant values by \( \pi_3(M) \).
Make a table of homotopy groups:

<table>
<thead>
<tr>
<th>OPS</th>
<th>$\pi_1$</th>
<th>$\pi_2$</th>
<th>$\pi_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^1$</td>
<td>$\mathbb{Z}$</td>
<td>${e}$</td>
<td>${e}$</td>
</tr>
<tr>
<td>$\mathbb{R}P^2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>$SO(3) = \mathbb{R}P^3$</td>
<td>$\mathbb{Z}_2$</td>
<td>${e}$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>$S^2$</td>
<td>${e}$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
</tr>
</tbody>
</table>

For example, a unit vector field $\hat{n}(\vec{x})$ (OPS = $S^2$) can possess point defects (characterized by an integer) and field configurations that are with asymptotically constant (also characterized by an integer), but no line defects (vortices).

Now start on manifold theory. Most of the spaces that occur in physics are differentiable manifolds. The idea is that a differentiable manifold is a topological space with enough extra structure to talk about differentiability, i.e., one can do calculus on manifolds.

**Def.** A differentiable manifold $M$ is a topological space $\hat{\Sigma}$ plus a set $\{U_i, \phi_i\}$, where $\{U_i\}$ is an open cover of $M$ (each $U_i$ is an open subset of $M$, and $\bigcup U_i = M$), and where $\phi_i$ is a map $\phi_i : U_i \to \mathbb{R}^n$ such that:

a) $\phi_i$ is a homeomorphism onto its image $V_i$ in $\mathbb{R}^n$