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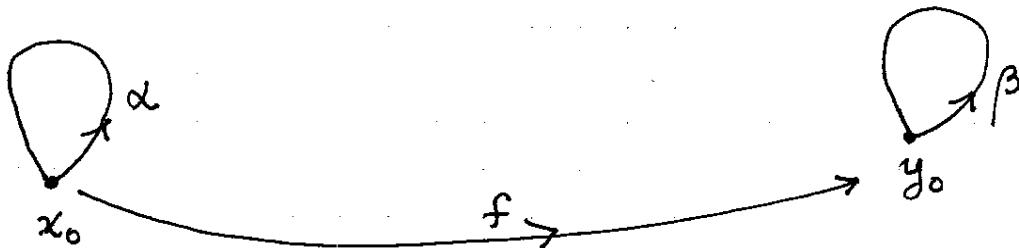
Return to the theorem: if X, Y are of the same homotopy type, then

$$\pi_1(X, x_0) = \pi_1(Y, y_0)$$

where $y_0 = f(x_0)$ and f is the map: $X \rightarrow Y$ that enters into the definition of "same homotopy type". Here "=" means, "is isomorphic to." Thus, if X, Y are connected, so that the homotopy groups are independent of base point, then $\pi_1(X) = \pi_1(Y)$.

Here is the beginning of a proof of the theorem.

Choose $x_0 \in X$, let $y_0 = f(x_0) \in Y$, and let $\alpha: I \rightarrow X$ be a loop at x_0 . Let β be the image of α under f ,



that is, $\beta = f \circ \alpha$, $\beta: I \rightarrow Y$ is a loop based at y_0 . Thus, f maps loops based at x_0 into loops based at y_0 .

What does f do to equivalence classes of loops? Let α, α' be two loops based at x_0 , and let $\alpha \sim \alpha'$ (α, α' are homotopic). Also let $\beta = f \circ \alpha$, $\beta' = f \circ \alpha'$. Are β, β' homotopic?



continuous

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Yes, because the deformation of α to α' produces a family of curves α_t , $\alpha_0 = \alpha$, $\alpha_1 = \alpha'$, and $f \circ \alpha_t = \beta_t$ is a deformation of β into β' . That is, the composition of two continuous maps is continuous: Here we compose f with the homotopy deforming α to α' .

This means that f induces a mapping between equivalence classes of curves based at x_0 into those based at y_0 . Call this mapping K :

$$K: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0).$$

Every equiv. class at x_0 corresponds to some equivalence class at y_0 , i.e. $K[\alpha] = [\beta]$.

The map K is injective. That means that if $[\alpha] \neq [\alpha']$, then $[\beta] \neq [\beta']$, i.e., if $\alpha \not\sim \alpha'$ then $\beta \not\sim \beta'$. That is, the only way we can have $[\beta] = [\beta']$ is if $[\alpha] = [\alpha']$. To prove this we may show that $[\beta] = [\beta'] \Rightarrow [\alpha] = [\alpha']$. Since above we showed that $\cancel{[\alpha]} = [\alpha'] \Rightarrow [\beta] = [\beta']$, the result will be $[\alpha] = [\alpha'] \Leftrightarrow [\beta] = [\beta']$, and K is injective.

So again assume we have 2 loops α, α' at $x_0 \in X$, and define $\beta = f \circ \alpha$, $\beta' = f \circ \alpha'$, so β, β' are loops at $y_0 = f(x_0) \in Y$. Now, however, assume $\beta \sim \beta'$. We wish to prove that $\alpha \sim \alpha'$.

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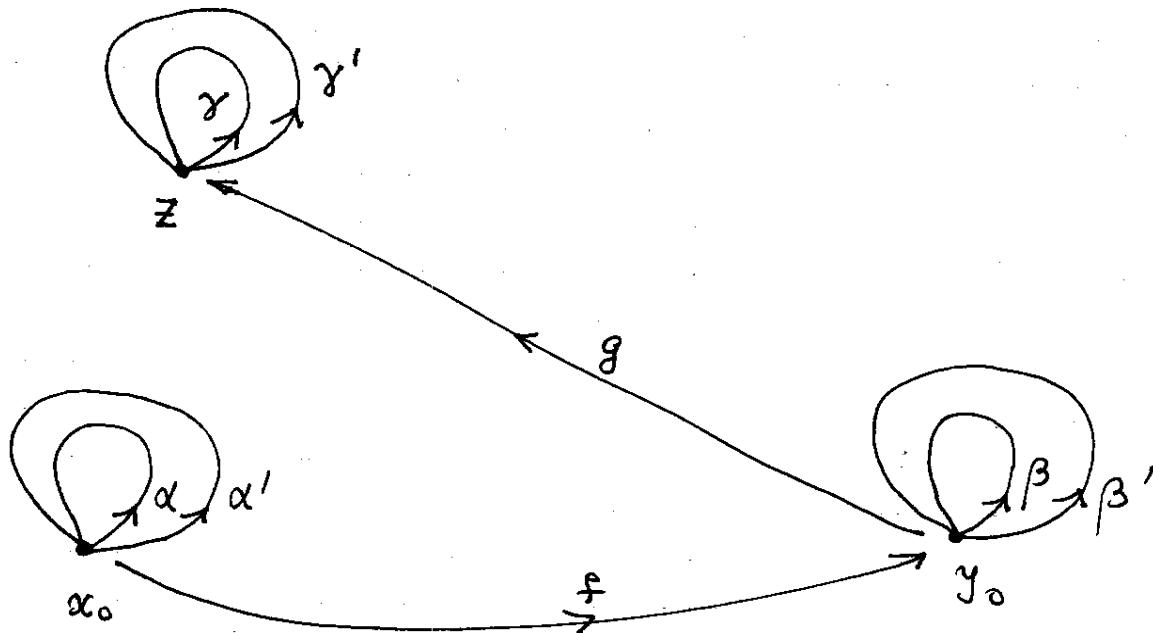
Use the map g to map β, β' back onto X , call the new loops γ, γ' :

$$\gamma = g \circ \beta = g \circ f \circ \alpha$$

$$\gamma' = g \circ \beta' = g \circ f \circ \alpha'$$

Since g is continuous, and since $\beta \sim \beta'$, we have $\gamma \sim \gamma'$. But γ, γ' are not based at x_0 , they are based at a new point call it $z \in X$,

$$z = g(y_0) = (g \circ f)(x_0).$$



We know that $\pi_1(X, z)$ and $\pi_1(X, x_0)$ are isomorphic if we can find a curve connecting z to x_0 . In fact, we can, since we know that $g \circ f \sim \text{id}_X$. Thus there exists a smooth family of maps $M_t : X \rightarrow X$, $t \in [0, 1]$, such that $M_0 = g \circ f$ and $M_1 = \text{id}_X$. Define a path $\eta : [0, 1] \rightarrow X$

by $\eta(t) = M_t x_0$. Then $\eta(0) = z$, $\eta(1) = x_0$: (4)
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And let η^{-1} run from x_0 to z .

Then

$$\eta^{-1} * \gamma * \eta = \delta$$

$$\text{and } \eta^{-1} * \gamma' * \eta = \delta'$$

are loops based at x_0 .

In fact, $\delta \sim \delta'$. We see this since we know $\gamma \sim \gamma'$, and if in the definitions of δ, δ'

we let γ deform into γ' , we get δ deforming into δ' . It's clear from the picture, too.

Now write out δ, δ' in another way:

$$\delta = \eta^{-1} * (g \circ f \circ \alpha) * \eta^* = \eta^{-1} * (M_0 \circ \alpha) * \eta$$

$$\delta' = \eta^{-1} * (g \circ f \circ \alpha') * \eta = \eta^{-1} * (M_0 * \alpha') * \eta$$

Now let $z(t) = M_t(x_0)$, so $z(t)$ runs along η as t goes from 0 to 1. Also let η_t be the curve that starts at $z(t)$ and ends at x_0 , following η , i.e., let

$$\eta_t : [0, 1] \rightarrow X, \quad \eta_t(s) = \eta(t + (1-t)s),$$

so $\eta_t(0) = z(t)$, $\eta_t(1) = x_0$. ~~closed~~ Thus $\eta_0 = \eta$,

and η_1 is the ^{constant} path at x_0 (it shrinks to x_0 as $t \rightarrow 1$).

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Now define

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$$\delta_t = \eta_t^{-1} * (M_t \circ \alpha) * \eta_t$$

$$\delta'_t = \eta_t^{-1} * (M_t \circ \alpha') * \eta_t$$

at $t=0$ we have $\delta_0 = \delta$, $\delta'_0 = \delta'$, at $t=1$ we have $\delta_1 = \alpha$, $\delta'_1 = \alpha'$. Thus, $\delta \sim \alpha$ and $\delta' \sim \alpha'$. But since $\delta \sim \delta'$ we have $\alpha \sim \alpha'$. QED for this part, $\beta \sim \beta' \Rightarrow \alpha = \alpha'$.

Thus $K: \pi_1(X \rightarrow x_0) \xrightarrow{\cong} \pi_1(Y, y_0)$ is injective.

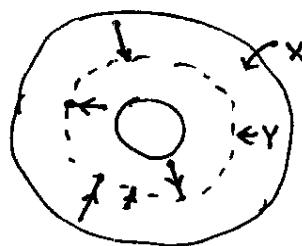
The remaining steps of the proof are to show that K is surjective (hence bijective), and then finally that it is a homomorphism (hence an isomorphism).

(Back to deformation retracts.)

Picture, example:

 $X = \text{annulus}$
 $Y = \text{circle (dotted)}$

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shown are paths of a point $x \in X$ under the map, as deformation parameter goes from 0 to 1.

official definition: Let X, Y be topological spaces, $Y \subset X$.

A deformation retract is a map $F: X \times [0,1] \rightarrow X$ such that

$$\begin{array}{l|l} F(x,0) = x & (\text{identity at } t=0) \\ \hline F(x,1) \in Y & (\text{into } Y \text{ at } t=1) \end{array} \quad \left| \quad \begin{array}{l} F(y,t) = y, \quad \forall t \in [0,1] \\ (\text{Y invariant, all } t). \end{array} \right.$$

Fact: If ~~X is~~ Y is a deformation retract of X , then X, Y are of same homotopy type. [$f: X \rightarrow Y$ is the retraction at $t=1$, $g: Y \rightarrow X$ is the inclusion.]

Another Def. If $x_0 \in X$ is the deformation retract of X (special case $Y = \{x_0\}$ = one point), then X is contractible. A contractible space is necessarily connected.
Follows immediately,

Corollary: If X is contractible, then $\pi_1(X) = \{e\}$ (the trivial group).

Def. If $\pi_1(X) = \{e\}$, then X is simply connected.

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Before going on, let's get some examples of fundamental groups, 9/18/08 obtained by intuition if nothing else.

- (1) First, $\pi_1(\mathbb{R}^n) = \{\text{id}\}$ (the trivial group), because all loops are contractible (the space is simply connected). This is obvious by drawing pictures,



- (1a) Note the special case $n=0$, $\mathbb{R}^0 = \text{one point} = \{0\}$, $\pi_1(\text{one point}_{\text{space}}) = \{\text{id}\}$.

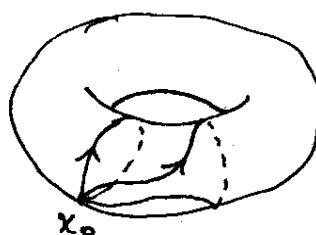
- (2) Next, $\pi_1(S^1) = \mathbb{Z}$, where the integer $n \in \mathbb{Z}$ is the "winding number" of the map. This is intuitively clear if you wrap a rubber band around a cylinder. We consider a more formal argument below.

- (3) Next, $\pi_1(S^n) = \{\text{id}\}$ for $n > 1$. This is intuitively obvious for S^2 (any closed loop on S^2 can be contracted to a point):



and a similar logic works for S^n for higher n .

- (4) Next, $\pi_1(T^2) = \mathbb{Z}^2$ (the 2-torus), as is intuitively clear by drawing pictures,



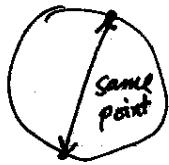
The homotopy class is determined by the ~~two~~ two "winding numbers".

This can be proved from the case $\pi_1(S^1) = \mathbb{Z}$ by using the theorem 9/18/08 that the fundamental group of the Cartesian product of two arcwise connected spaces is the Cartesian products of the groups,

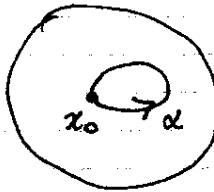
$$\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y).$$

The same thing shows that $\pi_1(T^n) = \mathbb{Z}^n$ (n winding numbers on an n -torus), since $T^n = S^1 \times \dots \times S^1$ (n times).
(also meaton cylinder = $\mathbb{R} \times S^1$).

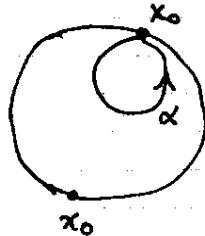
- (5) Take the case of \mathbb{RP}^2 , ~~disk~~ with opposite bdry points identified.



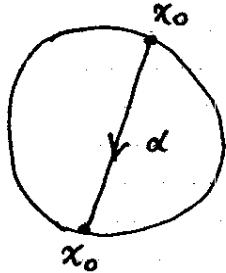
Easy to find contractible loops:



or



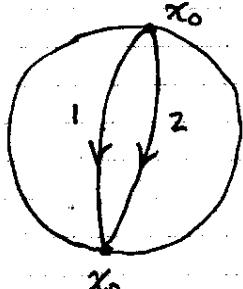
Here is a ~~curve~~ ^{loop} that is not contractible:



Because you can't bring the two attachment points together (they stay on opposite sides of the bdry under any contin. deformation).

So $[\alpha]$ is a nontrivial element of $\pi_1(\mathbb{RP}^2, x_0)$. Now look at $[\alpha] * [\alpha]$, equiv. class of loops that ~~are~~ are homotopic to traversing α twice.

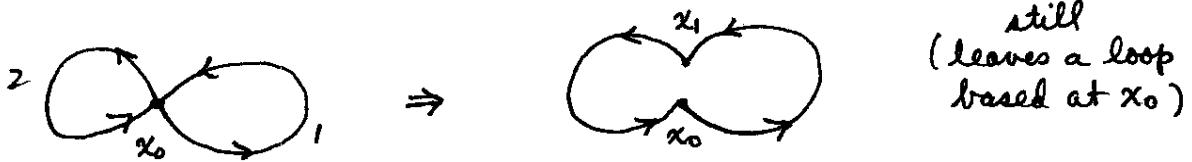
Label the 2 traversals 1, 2 to indicate order, and bow them out slightly to separate them:



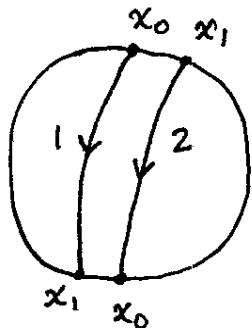
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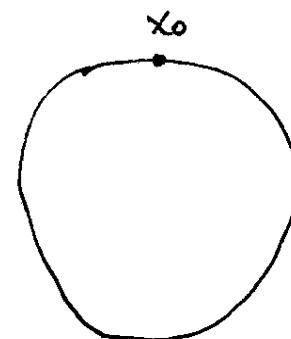
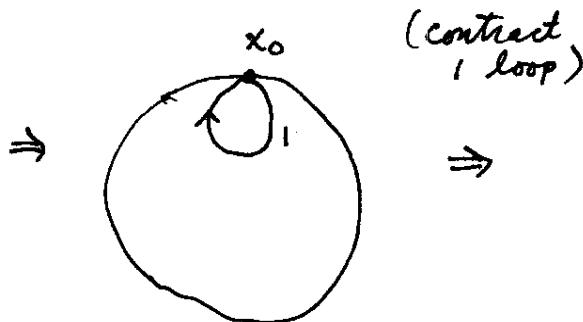
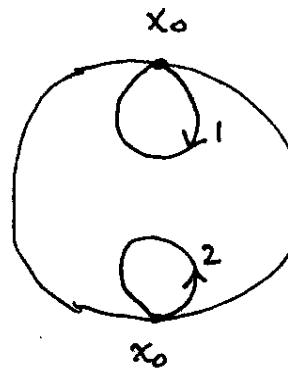
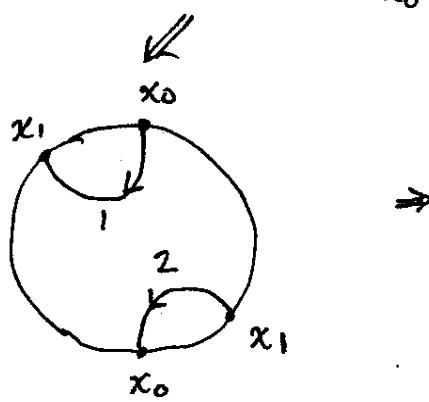
This path starts at x_0 , passes through x_0 a 2nd time, then comes back to x_0 a 3rd time. Pull the path away from x_0 on the second encounter, like this:



which on \mathbb{RP}^2 looks like this:



Then deform further by moving x_1 around to meet x_0 :



Thus $[\alpha] * [\alpha] = [c]$ = trivial class, contractible.

Hence $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$. A similar argument shows that

$\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$, $n \geq 2$.

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This \mathbb{Z}_2 is responsible for the line defect (vortex) in a nematic liquid, that can annihilate when it meets another such line defect, leaving behind no defect at all. [It's not that vortices of "opposite charge" are annihilating, rather, there is only one charge and it obeys the rule $1+1=0$.] 9/18/08

While on the subject of \mathbb{RP}^n , note special case $n=1$. Recall, \mathbb{RP}^n is the sphere S^n with antipodal points identified; equivalently, it is the n -dimensional disk D^n (= the "northern hemisphere" of S^n) with antipodal points on the boundary identified. (The disk D^n is region $r \leq 1$ in n -dim. space \mathbb{R}^n ; it is the sphere S^{n-1} plus all interior points.) So, for $n=1$, we get a circle with opposite points identified, or D^1 , the 1-disk, which is a line segment with ~~opposite~~ endpoints identified:

$$\mathbb{RP}^1 = \frac{\text{circle}}{S^1} = \frac{\text{half-circle}}{\frac{D^1}{\sim}} = \frac{\text{circle}}{\sim} = S^1$$

You might say the final circle is $1/2$ as big as the first one.

Now is a good time to comment on the relationship between classical rotations and spin rotations in QM. A classical rotation is an element of $SO(3)$, a linear map of \mathbb{R}^3 onto itself that preserves lengths, angles, and cross products. It takes 3 parameters to specify some $R \in SO(3)$ (i.e., $SO(3)$ is a 3-dimensional manifold). These parameters can be specified in various ways. One is the Euler angles (often an ugly choice). Another is the axis-angle parameterization:

(continued on p. 11).

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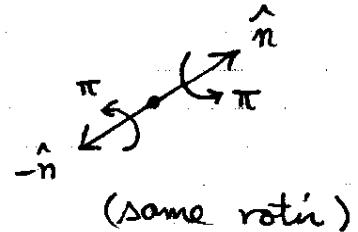
Take it as geometrically obvious that an arbitrary (proper) rotation can be written in axis-angle form:

$$\hat{n} \in S^2, \quad 0 \leq \theta \leq \pi$$

$$R(\hat{n}, \theta) = \begin{array}{c} \nearrow \downarrow \\ \text{using right hand rule.} \end{array}$$

The parameterization is unique except when $\theta=0$, where $R(\hat{n}, 0) = I$ for any \hat{n} , and at $\theta=\pi$, where

$$R(\hat{n}, \pi) = R(-\hat{n}, \pi)$$



So if we write $\vec{\theta} = \hat{n}\theta$, so that $\vec{\theta} \in \mathbb{R}^3$, then $SO(3)$ is identified with a sphere (the 3D, solid interior of a sphere in \mathbb{R}^3) out to a radius of π , including the surface (S^2) at $\theta=\pi$, but with antipodal points $(\hat{n}, -\hat{n})$ on the surface identified. In other words,

$$SO(3) = \mathbb{RP}^3.$$

This is an example of a group manifold.

As for $SU(2)$, physically, spin rotations

it is the set of 2×2 , complex, unitary matrices

with $\det = +1$:

$$U \in SU(2) \Rightarrow UU^+ = U^+U = I$$

and $\det U = +1$.

The condition $UU^+ = U^+U = I$ means that the rows and columns form pairs of orthonormal, complex, unit vectors (in \mathbb{C}^2). Write

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad a, b, c, d \in \mathbb{C},$$

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Because of the conditions $U^+U = I$, $\det U = +1$, the 4 complex components of U satisfy certain constraints, and U can be written in terms of 4 real parameters $(x_0, x_1, x_2, x_3) \in \mathbb{R}^4$

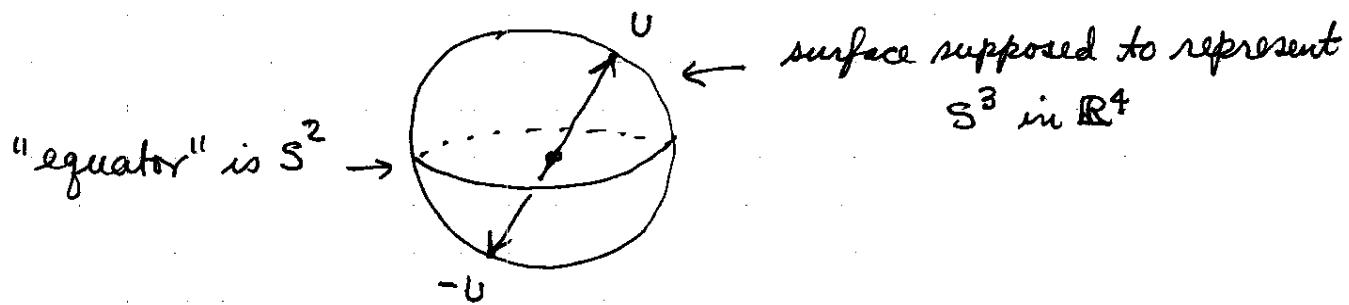
$$U = x_0 I - i \vec{x} \cdot \vec{\sigma} = \begin{pmatrix} x_0 - ix_3 & -x_2 - ix_1 \\ x_2 - ix_1 & x_0 + ix_3 \end{pmatrix} \quad \hookrightarrow \equiv (x_0, \vec{x})$$

$\mathfrak{su}(2)$

where $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$. The (x_0, x_1, x_2, x_3) are the Cayley-Klein parameters, and they show that topologically,

$$\mathfrak{su}(2) = S^3.$$

The relation between $\mathfrak{su}(2) = S^3$ and $\mathfrak{so}(3) = \mathbb{RP}^3$ is just the identification of antipodal points U and $-U$ in S^3 with a single element $R \in \mathfrak{so}(3) = \mathbb{RP}^3$.



$\mathfrak{so}(3)$ can be thought of as the "northern hemisphere" with antipodal points on the "equator" (S^2) identified. This is the solid ball picture of $\mathfrak{so}(3)$ (in $\vec{\theta}$ coordinates).

(B)

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Motivation for studying relationship between $SO(3)$ and $SU(2)$. Consider evolution of spin $\frac{1}{2}$ particle in magnetic field $\vec{B} = \vec{B}(t)$ which we allow to be time-dep. Define

$$\vec{\omega}(t) = g \frac{e}{2mc} \vec{B}(t)$$

a vector with dimensions of frequency ($g = g$ -factor of particle).

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{C}^2$ be the usual spinor. The Schrödinger eqn is

$$i\hbar \frac{dX}{dt} = \vec{\omega}(t) \cdot \left(\frac{\hbar}{2}\vec{\sigma}\right) X \quad (\text{Qu})$$

where $\frac{\hbar}{2}\vec{\sigma}$ is the spin operator. Let $\vec{S}(t)$ be the expectation value of the spin operator,

$$\vec{S}(t) = \langle X(t) | \frac{\hbar}{2}\vec{\sigma} | X(t) \rangle \quad (\text{Class})$$

so that \vec{S} is a c-number vector (not a vector of operators, $\vec{S} \in \mathbb{R}^3$).

Then

$$\frac{d\vec{S}}{dt} = \vec{\omega}(t) \times \vec{S} \quad (\text{Cl}).$$

(Qu) is the "quantum eqn" and (Cl) is the "classical" eqn. (classical in the sense that eqns just like this occur in classical mechanics, they are the Euler equations). The solutions of (Qu) and (Cl) are

$$X(t) = U(t) X_0, \quad U(t) \in SU(2)$$

$$\vec{S}(t) = R(t) \vec{S}_0, \quad R(t) \in SO(3)$$

where $U(0) = 1$ (the 2×2 identity) and $R(t) = I$ (the 3×3 identity).

The functions $U(t)$ and $R(t)$ are actually paths on the group manifolds $SU(2)$ and $SO(3)$. Let

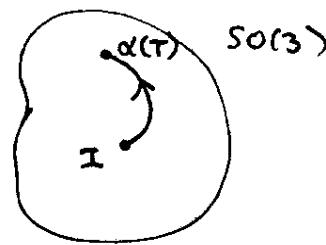
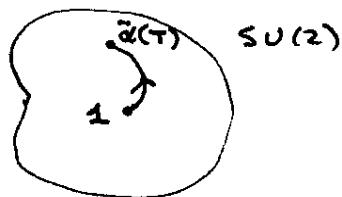
($T = \text{final time}$)

$$\alpha: [0, T] \rightarrow SO(3)$$

$$\bar{\alpha}: [0, T] \rightarrow SU(2)$$

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be two paths in $SO(3)$ and $SU(2)$, where $\alpha(t)$ means $R(t)$ and $\bar{\alpha}(t)$ means $U(t)$, satisfying $\bar{\alpha}(0) = I$, $\alpha(0) = I$. Picture on the group manifolds,



Consider the stat: "If you rotate a neutron by 360° , it doesn't return to its original self but rather undergoes a phase change of -1 . You have to rotate it by 720° to make it return to itself." Actually it is not the final value of the classical rotation $R(t)$ (or $\alpha(t)$) that determines the outcome, but rather the history. Here is a correct stat:

Let $R(T) = \alpha(T) = I$ (at $t=T$). Then $\alpha: [0, T] \rightarrow SO(3)$ is a loop based at I . But $SO(3) = \mathbb{RP}^3$ (topologically speaking), so there are two homotopy classes the loop α can be in, the trivial class or the nontrivial class, since $\pi_1(\mathbb{RP}^3) = \mathbb{Z}_2$. Then

$$U(T) = \bar{\alpha}(T) = \begin{cases} +1 & \text{if } \alpha \in \text{trivial (contractible) class} \\ -1 & \text{if } \alpha \in \text{other class.} \end{cases}$$

The final state of the neutron depends on the homotopy class of the loop α in $SO(3)$. In fact one may say that the existence of spin is related to this nontrivial homotopy group $\pi_1(SO(3)) = \mathbb{Z}_2$.

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There is an important map $p: \text{SU}(2) \rightarrow \text{SO}(3)$ that occurs in this theory. (p stands for "projection.") It is defined by ...

$$R_{ij} = \frac{1}{2} \text{tr} (U^+ \sigma_i U \sigma_j) \quad \text{where } U \in \text{SU}(2),$$

i.e., it defines a function $R(U)$ or $R = p(U)$. One can show that

$$R(t) = p(U(t))$$

in the spin problem, i.e., p maps the path $\bar{\alpha}(t)$ in $\text{SU}(2)$ into $\alpha(t)$ in $\text{SO}(3)$. Note that $p(U) = p(-U)$, so the inverse $p^{-1}(R)$ of $R \in \text{SO}(3)$ consists of 2 points U and $-U$ (it turns out there are only these two). p is a two-to-one projection.

$\text{SU}(2)$ is ~~an example~~ said to be a double cover of $\text{SO}(3)$. This is an example of a space M ($\text{SO}(3)$) and its covering space \bar{M} (here $\text{SU}(2)$). The projection p in the general case ~~is~~ is a map $p: \bar{M} \rightarrow M$ from the covering to the covered spaces. The path $\bar{\alpha}(t)$ defined above in $\bar{M} = \text{SU}(2)$ is called the lift of the path $\alpha(t) = R(t)$ in $M = \text{SO}(3)$. We mention all this (as yet) undefined terminology to give an example a preview of what will come.

Covering spaces don't have to be groups, but in this example they are, and there is extra structure because of that. For example, $p: \text{SU}(2) \rightarrow \text{SO}(3)$ is a group homomorphism, with kernel $\{1, -1\}$ (the image is all of $\text{SO}(3)$).