

9/16/08

①

The dimension of the free part of  $H_r(K, \mathbb{Z})$ , which is the dimension of  $H_r(K, \mathbb{R})$ , is called the  $r$ -th Betti number of  $K$ , denoted

$$b_r(K) = \dim H_r(K, \mathbb{R}).$$

There is an interesting relation between the Euler characteristic and the Betti numbers.

The usual Euler characteristic on a 2D surface  $\overset{s}{\lambda}$  (sphere, torus) is

$$\chi(s) = f - e + v,$$

$$f = \# \text{faces} \quad (\# \text{2-simplexes in } K) \approx$$

$$e = \# \text{edges} \quad (\# \text{1-simplexes in } K)$$

$$v = \# \text{vertices} \quad (\# \text{0-simplexes in } K).$$

On the sphere  $S^2$ ,  $\chi = 2$ , independent of the triangulation. The obvious generalization of this is to replace  $f, e, v$  etc. by the dimension of the  $r$ -th chain group  $C_r(K, \mathbb{R})$ , which is just the number of independent  $r$ -chains in  $K$ . So, define

$$\chi(K) = \sum_{r=0}^n (-1)^r \dim C_r(K, \mathbb{R}). \quad (n = \dim K).$$

(These dimensions are not topological invariants.)

Now,  $\partial_r$  is a map:  $C_r(K) \rightarrow C_{r-1}(K)$ , so

(under  $\mathbb{R}$ , not  $\mathbb{Z}$ , so  $\partial$  is a linear map),

$$\dim \ker \partial_r + \dim \text{im } \partial_r = \dim C_r(K, \mathbb{R}).$$

$$\text{or } \dim Z_r(K) + \dim B_{r-1}(K) = \dim C_r(K)$$

(over  $\mathbb{R}$  understood now.)

In the case  $r=0$ , there is no  $B_{-1}(K)$ , but  $Z_0 = C_0$  so we may interpret  $\dim B_{-1}(K)$  as zero.

9/16/08

(2)

So,

$$\chi(K) = \sum_{r=0}^n (-1)^r [\dim Z_r(K) + \dim B_{r-1}(K)].$$

$$\begin{aligned}
 \text{2nd sum} &= \sum_{r=0}^n (-1)^r \dim B_{r-1}(K) \\
 &= \sum_{r=1}^n (-1)^r \dim B_{r-1}(K) \quad (\text{drop } B_0 \text{ term}) \\
 &= - \sum_{r=0}^{n-1} (-1)^r \dim B_r(K) \quad (\text{shift index}) \\
 &= - \sum_{r=0}^n (-1)^r \dim B_r(K) \quad (\text{add } r=n \text{ term, } \dim B_n(K)=0).
 \end{aligned}$$

So whole sum becomes,

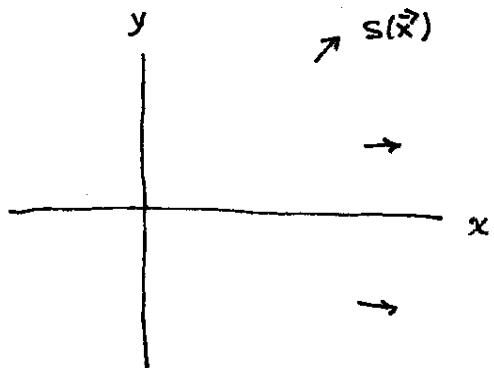
$$\begin{aligned}
 \chi(K) &= \sum_{r=0}^n (-1)^r [\dim Z_r(K) - \dim B_r(K)] \\
 &= \sum_{r=0}^n (-1)^r \dim H_r(K) = \sum_{r=0}^n (-1)^r b_r(K).
 \end{aligned}$$

This is the Euler-Poincaré thm, relating the Euler-characteristic to the Betti numbers, manifest topological invariants.

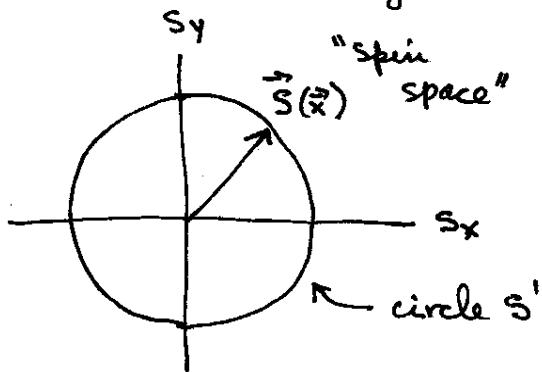
Next we turn to homotopy theory.

9/16/08 (3)

Begin with motivation for homotopy theory from CM physics. (But  $\exists$  many other applications.) Consider a 2D model of a Heisenberg ferromagnet, treated as a continuum, so we have a spin vector  $\vec{s}(\vec{x})$  whose magnitude is fixed,  $|\vec{s}| = S = \text{const.}$  Assume  $\vec{s}$  lies in the plane.

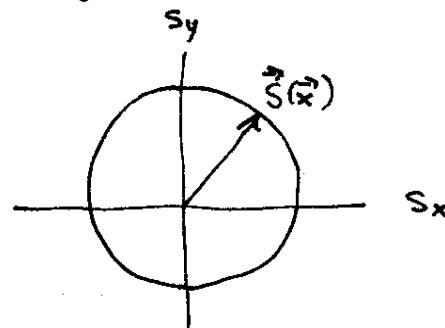
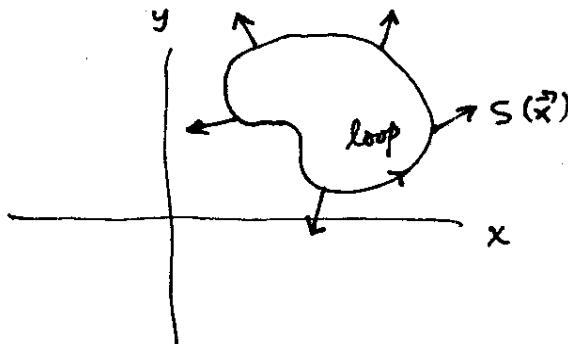


Can regard the spin field as a map  $f: X \rightarrow S^1$ , where  $X$  is the region of the plane occupied by the ferromagnet (maybe  $X = \text{all of } \mathbb{R}^2$ , depends on the model). The range is  $S^1$  (the circle) because only the direction of  $\vec{s}$  can vary, not the magnitude.



The same picture would apply to any 2D vector field in the plane (or some region of it), as long as the vector field did not vanish, (that is, you could look at the direction of the vector field).

Now consider a closed loop in the plane. As we go around the loop, the vector  $\vec{s}(\vec{x})$  goes around its circle, returning to where it started.



The number of times  $\vec{s}(\vec{x})$  goes around its circle as  $\vec{x}$  goes around its loop is the winding number of the field around the loop.

9/16/08 (4)

Note that if  $\vec{S}(\vec{x}) = \text{const.}$  (indep. of  $\vec{x}$ ) then the winding # is 0.

Now we can prove that if the winding # is not 0, then there must be a singularity inside the loop. Suppose not. Then we can continuously contract the loop until it is very small. Since we are assuming  $\vec{S}$  is continuous (this is what we mean by singularity-free) then over a small loop  $\vec{S}$  is nearly constant and the winding # is zero. But the winding number is an integer and cannot change discontinuously. So there must be a discontinuity inside the loop.

This would be called a point defect in the plane in CM terminology. The loop provides a map  $f: S' \rightarrow S'$ , where the first  $S'$  is the loop in the plane and the second is in spin space. The winding number is a characteristic of the map of the circle to itself, also called the Brouwer degree of the map. This number, being an integer, cannot change under continuous deformations of either the loop or the field  $\vec{S}(\vec{x})$  itself. Such changes can be thought of as inducing continuous changes of the map  $f: S' \rightarrow S'$ . Homotopy theory studies invariants of maps such as this one under continuous changes.

The range of  $f$  ( $S'$  in this case) is called "order parameter space" in CM applications. Another problem with the same order parameter space is superfluid  $^4\text{He}$ , where the superfluid is described by a field,

$$\Psi(\vec{x}) = A e^{i\varphi}$$

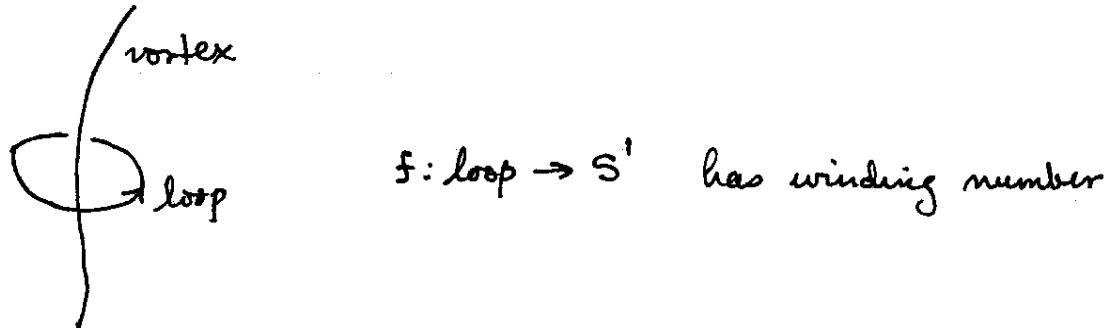
c-number

( $\varphi$  is an expectation value of a quantum field, thus it is a classical, complex field on  $\mathbb{R}^3$  or some domain  $X \subset \mathbb{R}^3$ ).

9/16/08

The amplitude  $A$  is related to the superfluid density which normally is nonzero throughout the volume. The phase  $\varphi$  is related to the fluid velocity via  $\vec{v} = \frac{i}{m} \nabla \varphi$ .

The phase specifies a point on a circle  $e^{i\varphi}$  in the complex plane, so again we may regard order parameter space as a circle and think of a map  $f: X \rightarrow S^1$  where now  $X \subset \mathbb{R}^3$ . Or if we restrict  $f$  to a loop in  $X$  we get a map  $f: S^1 \rightarrow S^1$  as before, with a winding number. Again, the logic shows that if the winding number is nonzero, then the ~~closed~~ loop must enclose a singularity. This time, however, the domain is in  $\mathbb{R}^3$  so the singularity is a line, not a point. It is a line defect, in this case, called a vortex.



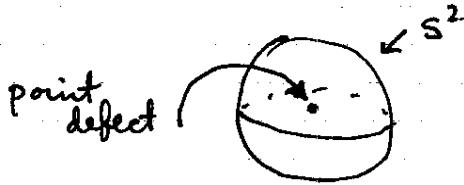
A similar situation occurs in superconductors, when the Cooper-paired spins are in the singlet state (hence  $\psi$  describes a spin 0 boson, like  ${}^4\text{He}$ ).

A different type of order parameter space occurs with nematic liquids. These are liquids with long molecules that behave like rigid rods, which tend to align with their neighbors. The alignment is not described by a vector field, however, because the properties of the nematic are the same in both directions, like a double-headed object  $\leftrightarrow$ .

9/16/08

(6)

Thus the order parameter space is  $\mathbb{R}\mathbb{P}^2$ . Nematics support line defects that are studied by examining continuous deformations of maps  $f: S^1 \rightarrow \mathbb{R}\mathbb{P}^2$  ( $S^1$  because a circle goes around a line defect). It turns out that nematics also support point defects that are studied by surrounding the point with a <sup>closed</sup> surface, i.e., one homeomorphic to  $S^2$ , and studying invariants of maps  $f: S^2 \rightarrow \mathbb{R}\mathbb{P}^2$ .



There are other kinds of order parameter spaces. For spins in 3D, the spin vector is identified with a point on  $S^2$ . There are no line defects, but point defects are possible and lead us to study maps:  $S^2 \rightarrow S^2$ . Another example is superfluid  ${}^3\text{He}$  in the dipole locked phase, where the order parameter can be thought of as two vectors (nonzero), locked to be orthogonal  $\uparrow$  to one another. The orientation of such an object is specified relative to a standard orientation, by means of a rotation  $\in SO(3)$ . Hence order parameter space is  $SO(3)$ , which is homeo. to  $\mathbb{R}\mathbb{P}^3$  (as it turns out).

To study defects we have looked at maps from  $S^1$  or  $S^2$  to order parameter space ( $S^1, \mathbb{R}\mathbb{P}^2, S^2, SO(3), \dots$ ). A map from  $S^3$  to order parameter space occurs when we want to study singularity-free field configurations that in  $\mathbb{R}^3$  that satisfy the constraint of having a fixed asymptotic value as  $r \rightarrow \infty$ . We just "compactify"  $\mathbb{R}^3$  into  $S^3$  by introducing the "point at  $\infty$ " (where the field has a fixed value), and ~~so~~ study maps  $f: S^3 \rightarrow \text{O.P.S.}$  such that  $f(\text{pt. at } \infty) = \text{given value}$ . For example, the constant map

9/16/08 ⑦

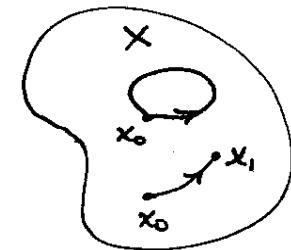
is one such field configuration, but (depending on the O.P.S.) there may be others which cannot be reached by continuous deformation from the constant case. Again we are interested in properties of maps  $f: S^n \rightarrow M$  (some manifold = O.P.S.) that are invariant under contin. deformations.

We take the preceding as motivation for considering continuous deformations of maps:  $S^n \rightarrow M = \text{some manifold}$ . First take case  $S^1$ .

Let  $I = [0, 1]$ .

A path is a map  $f: I \rightarrow X = \text{some topological space}$

A loop is a path such that  $f(0) = f(1) = x_0 = \text{"base point"}$



Note, path can self-intersect, or even be just a point (the constant path).

Now, basic properties of paths and loops. First, we can multiply paths if the endpoints match. This is just cateration.

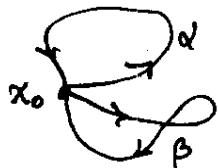


Let  $\alpha, \beta: [0, 1] \rightarrow X$ , let  $\alpha(1) = \beta(0)$ . Define:

$$(\alpha * \beta)(s) = \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}.$$

Notice order: traverse  $\alpha$  first,  $\beta$  second.

Note that loops based at a common point can always be multiplied.



$\alpha * \beta$  meaningful.

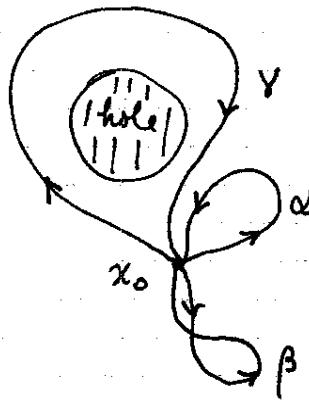
9/16/08

Def: let  $\alpha: [0,1] \rightarrow X$  be a path. Define the inverse path by

$$\alpha^{-1}: [0,1] \rightarrow X, \quad \alpha^{-1}(s) = \alpha(1-s).$$

The inverse path just traverses the original path in the reverse order.

Now we define homotopic equivalence, specializing to the case of ~~cont~~ loops based at a point. Idea:



$\alpha$  is homotopically equivalent to  $\beta$ , but not to  $\gamma$  (they cannot be continuously deformed into one another).

$$\downarrow = I$$

Let  $\alpha, \beta: [0,1] \rightarrow X$  be loops based at  $x_0 = \alpha(0) = \alpha(1) = \beta(0) = \beta(1)$ .

Then  $\alpha$  is said to be homotopic to  $\beta$  if  $\exists$  a continuous function

$F: I \times I \rightarrow X$  (think,  $F(s,t)$ ,  $s$  = parameter of path,  $t$  = deformation param.)

such that

$$F(s,0) = \alpha(s)$$

$$F(s,1) = \beta(s) \quad \text{and} \quad F(0,t) = F(1,t) = x_0.$$

Then  $F$  (the deformation map) is said to be the homotopy.

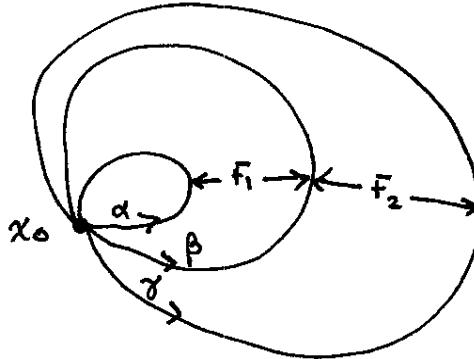
Basic fact is, "homotopic to" is an equivalence relation. To prove this, have to show 3 things:

(9)

9/16/08

- (a)  $\alpha \sim \alpha$
- (b)  $\alpha \sim \beta \Rightarrow \beta \sim \alpha$
- (c)  $\alpha \sim \beta$  and  $\beta \sim \gamma \Rightarrow \alpha \sim \gamma$ .

I will just prove (c), which is the hardest. A picture gives the idea:



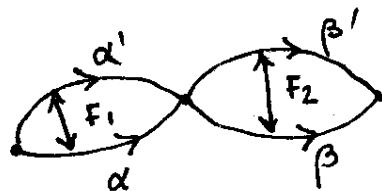
$$\text{Let } F(s, t) = \begin{cases} F_1(s, 2t), & 0 \leq t \leq 1/2 \\ F_2(s, 2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

Then  $F$  deforms  $\alpha$  into  $\gamma$ .

Thus we have equivalence classes of loops based at a point  $x_0$ ,  $[\alpha], [\beta]$ , etc. We want to define

$$[\alpha] * [\beta] = [\alpha * \beta] \quad (\text{multiplication of classes}).$$

But we have to show that this is consistent (independent of choice made for representative element). That is, let  $\alpha' \sim \alpha$  and  $\beta' \sim \beta$ . Want to show that  $\alpha' * \beta' \sim \alpha * \beta$ . Draw a picture, makes it obvious:



[Here picture drawn for arbitrary paths, but you get loops if endpoints are same point.]. Picture shows what to do:

(10)

Let  $F(s, t) = \begin{cases} F_1(2s, t), & 0 \leq s \leq \frac{1}{2} \\ F_2(2s-1, t), & \frac{1}{2} \leq s \leq 1 \end{cases}$

9/16/08

$F$  is the homotopy that deforms  $\alpha * \beta$  into  $\alpha * \gamma$ .

Turns out, this  $*$  law on equivalence classes of loops based at a point  $x_0$  defines a group. Need to show:

$$([\alpha][\beta])[\gamma] = [\alpha]([\beta][\gamma]) \quad \text{associative law} \quad (\text{drop } * \text{ now})$$

$$[\alpha][c] = [c][\alpha] = [\alpha], \quad [c] = \text{equiv. class of constant loop}$$

$$[\alpha^{-1}][\alpha] = [\alpha][\alpha^{-1}] = [c]. \quad c: [0, 1] \rightarrow X: s \mapsto x_0.$$

Thus  $[c]$  is the identity: It is the equivalence class of loops that can be contracted to a point.

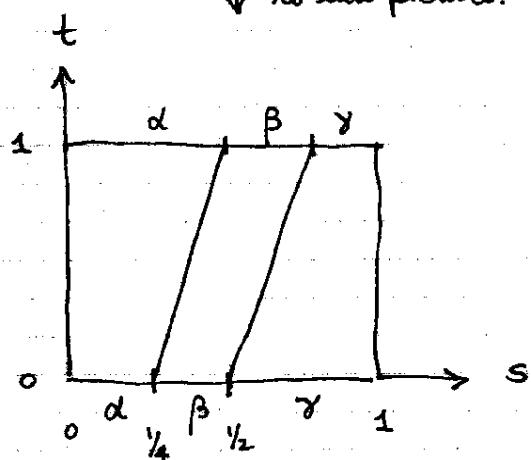
I will just prove the associative law (the others are easy). A picture makes it obvious, and also shows how to construct the formal proof:



$$(\alpha * \beta)(s) = \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

deform according  
to this picture.

$$((\alpha * \beta) * \gamma)(s) = \begin{cases} \alpha(4s), & 0 \leq s \leq \frac{1}{4} \\ \beta(4s-1), & \frac{1}{4} \leq s \leq \frac{1}{2} \\ \gamma(2s-1), & \frac{1}{2} \leq s \leq 1. \end{cases}$$



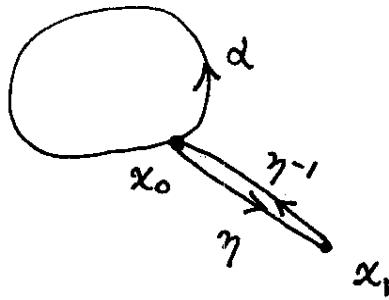
9/16/08

(11)

Thus, the set of equivalence classes of loops based at a point  $x_0$  constitutes a group, denoted

$$\pi_1(X, x_0) = \text{first homotopy group of } X \text{ based at } x_0.$$

This group is defined relative to a base point  $x_0$ . If we choose a different ~~go~~ base point we get a different group, say,  $\pi_1(X, x_1)$ . What is the relation between these? Answer is easy in an arc-wise connected space (which for us is just a connected space), in which a path  $\eta: [0, 1] \rightarrow X$ ,  $\eta(0) = x_0$ ,  $\eta(1) = x_1$ , always exists.



Given  $\alpha$  based at  $x_0$ , can create an  $\alpha'$  based at  $x_1$  by writing,

$$\alpha' = \eta^{-1} * \alpha * \eta.$$

Moreover, if  $\alpha \sim \beta$ , then  $\alpha' \sim \beta'$ , so the conjugation by  $\eta$  preserves the equivalence class structure. (Easy to show.) So we have a mapping parameterized by  $\eta$ ,

$$P_\eta: \pi_1(X, x_0) \rightarrow \pi_1(X, x_1): [\alpha] \mapsto [\eta^{-1} * \alpha * \eta]$$

In fact,  $P_\eta$  is a group isomorphism. We prove this by showing first that  $P_\eta$  is a homomorphism, then that  $P_\eta^{-1}$  is a homomorphism.

Proof that  $P_\eta$  is a homomorphism is easy.

9/16/08

(12)

Need to show that  $(P_\eta[\alpha])(P_\eta[\beta]) = P_\eta[\alpha \beta]$  (omit \*)

$$\text{LHS} = [\eta^{-1}\alpha\eta][\eta^{-1}\beta\eta] = [\eta^{-1}\alpha\eta\eta^{-1}\beta\eta] = [\eta^{-1}\alpha\beta\eta] = \text{RHS.}$$

Similarly, consider  $P_\eta^{-1} : \pi_1(x, x_1) \rightarrow \pi_1(x, x_0)$  defined by

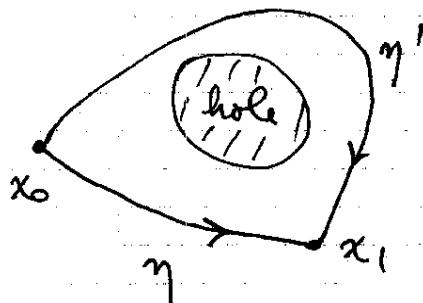
$$P_\eta^{-1}[\gamma] = [\eta\gamma\eta^{-1}] \quad \text{where } \gamma \in \pi_1(x, x_1) \text{ and } \eta \text{ as before.}$$

Then easily show that  $P_\eta^{-1}$  actually is the inverse of  $P_\eta$ . Thus,  $P_\eta$  is a bijection, hence an isomorphism.

Thus, while  $\pi_1(x, x_0)$  depends on the base point  $x_0$ , the group  $\pi_1(x, x_1)$  at any other point (remember this is an arcwise-connected space) is isomorphic to it. Thus, as abstract groups, these groups are the same: it is written simply as

$\pi_1(x)$ , called the fundamental group of  $X$ .

But notice, the isomorphism between  $\pi_1(x, x_0)$  and  $\pi_1(x, x_1)$  is not a natural isomorphism, because it depends on the choice of  $\eta$  connecting  $x_0$  and  $x_1$ .



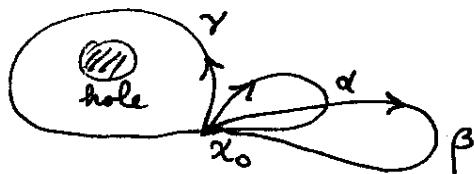
$P_\eta \neq P_{\eta'}$  in this picture.

9/16/08

(13)

Review: We are thinking of maps  $f: S^1 \rightarrow M$  of the circle onto a given space  $M$ , in order to understand the topology of  $M$  and for other reasons. We usually represent these maps by in a different form,  $f: I \rightarrow M$  where  $I = [0, 1]$  is the unit interval and where we require  $f(0) = f(1) = x_0 \in M$ . We call such a map a loop based at  $x_0$ .

Loops are considered equivalent (homotopic) if they can be continuously deformed into one another. The map that does the deformation is called the homotopy.



In the picture,  $\alpha \sim \beta$  but  $\alpha \not\sim \gamma$  (all loops based at  $x_0$ ).

The <sup>set of</sup> equivalence classes of loops based at a point  $x_0$  can be given a group structure, in which the multiplication is just catenation. The group is denoted  $\pi_1(M, x_0)$ , the first homotopy group based at  $x_0$ .

The group  $\pi_1(M, x_1)$  based at a different point is isomorphic to  $\pi_1(M, x_0)$ , if  $M$  is connected. Thus, as abstract groups, they are the same.

This group is denoted  $\pi_1(M)$ , the fundamental group.

9/16/08

(14)

Now we explore some of the properties of the fundamental group  $\pi_1(M)$  of a manifold  $M$ , and work on ways of computing it.

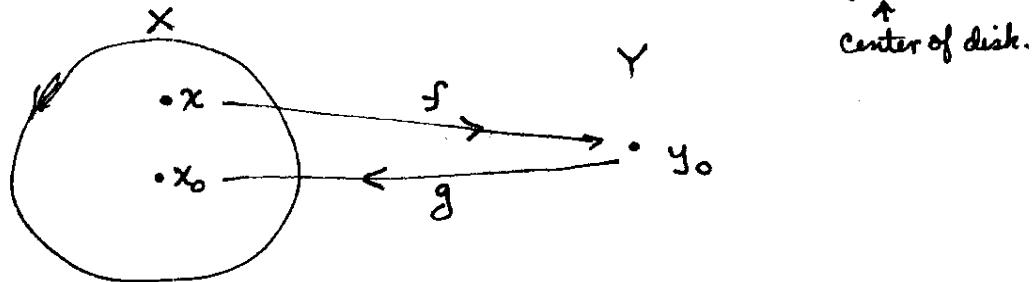
Recall defn of homeomorphism: let  $X, Y$  be topological spaces. Then ~~exist~~  
~~are homeomorphic~~  
are homeomorphic if  $\exists$  contin. maps  $f: X \rightarrow Y$ ,  $g: Y \rightarrow X$  such that  $f \circ g = \text{id}_Y$  and  $g \circ f = \text{id}_X$ .

Now we introduce a weaker concept. Let  $X, Y$  be topological spaces.

Def.  $X$  and  $Y$  are of the same homotopy type if  $\exists$  contin. maps  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$  such that  $f \circ g \sim \text{id}_Y$  and  $g \circ f \sim \text{id}_X$ . ( $\sim$  means, "is homotopic to"). Point is that = for a homeomorphism is replaced by  $\sim$  for same homotopy type.

An example of spaces that are of same homotopy type but not homeomorphic.

Let  $X = 2\text{-disk } D^2$ ,  $Y = \text{single point } y_0$ . Let  $f: X \rightarrow Y$  be the const. map  $f(x) = y_0$ ,  $\forall x \in X$ . Let  $g: Y \rightarrow X$  map  $y_0$  onto a certain point  $x_0 \in X$ .



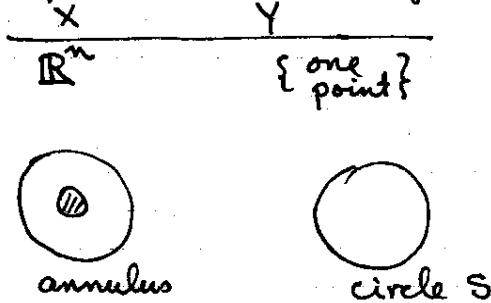
Not homeomorphic, because  $f$  not invertible. ~~Are~~ Are  $X, Y$  of same homotopy type? Well,  $f \circ g: Y \rightarrow Y: y_0 \mapsto y_0$  is just the identity map  $\text{id}_Y$  (has to be, since  $Y$  only has one point.). Next  $g \circ f: X \rightarrow X: x$  (any  $x$ )  $\mapsto x_0$ , it is the constant map. So is  $g \circ f \sim \text{id}_X$ ? Yes, just shrink disk by radial factor, to make disk  $\rightarrow$  central point.

Notice that with "same homotopy type" (as with "homeomorphic")

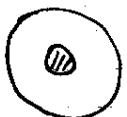
9/16/08 (15)

we don't talk about deforming spaces (which would require an imbedding space), we talk about deforming maps.

Other examples of same homotopy type spaces:



These examples show that being of "same homotopy type" allows dimensions of spaces  $X, Y$  to be different, no invertible map



annulus

circle  $S^1$ 

If sets are homeomorphic, then they are of same homotopy type, ~~and conversely this need not be true~~ (but not the converse).

Fact: "Same homotopy type" is an equivalence relation.

Thm: If  $X$  and  $Y$  are of the same homotopy type, then  $\pi_1(X) = \pi_1(Y)$ .

We won't prove this thm, but the proof is not very hard, and if you try it you will see the motivation for the definition of "same homotopy type". In any case, this theorem implies that homotopy groups are topological invariants.

So how to tell if spaces are of same homotopy type? May not be easy, but one case arises a lot in practice, occurs when  $Y$  is a subset of  $X$ .

Suppose we have a family of maps parameterized by a deformation parameter  $t \in [0,1]$  which is the identity at  $t=0$ , which maps  $X$  onto  $Y$  when  $t=1$ , and which leaves points of  $Y$  alone for all  $t$ .