Examples of Abelian groups include:

\[ \mathbb{R}^n, S', \mathbb{Z}, \mathbb{Z}_k \]

- Vector space \([e^0]\)
- Integers modulo \(k = \{0, 1, \ldots, k-1\}\).

We will only be interested in discrete Abelian groups, which excludes things like \(\mathbb{R}^n\) and \(S'\).

If \( G \) is a discrete Abelian group and \( x_1, \ldots, x_r \) are elements of \( G \) such that any \( g \in G \) can be written in the form,

\[ g = \sum_{i=1}^{r} n_i x_i, \quad n_i \in \mathbb{Z}, \]

then \( G \) is said to be generated by the \( \{x_i\} \) and the \( \{x_i\} \) are said to be the generators. If \( r < \infty \), then we say that \( G \) is finitely generated. For homology theory we only need finitely generated Abelian groups. Notice that so far we're not saying that the generators are minimal in number (and in any case they are not unique).

Either the group elements \( \sum_{i=1}^{r} n_i x_i \) for \( n_i \in \mathbb{Z} \) are all unique, or there is some duplication (some group elements can be represented as a "linear combination" of the generators in more than one way).

In the former case we say that the group is freely generated, and that \( G \) is a free Abelian group of rank \( r \). In this case, every \( g \in G \) can be uniquely represented as

\[ g = \sum_{i=1}^{r} n_i x_i, \]
and \( G \) is isomorphic to \( \mathbb{Z}^r \), \( G \cong \mathbb{Z}^r \), \( g \mapsto (n_1, \ldots, n_r) \).

In effect, there is only one free Abelian group of rank \( r \), it is \( \mathbb{Z}^r \), and it can be visualized as the integer lattice in \( r \)-dimensional \( \mathbb{R}^r \).

About the non-uniqueness of the generators. Take the case \( r = 2 \) for illustration. Represent the generators \( x_1, x_2 \) as unit vectors in the plane, and the group as the lattice.

The "basis" \((x_1, x_2)\) spans the lattice, but it is not unique. We could use

\[
\begin{align*}
y_1 &= 2x_1 - x_2 \\
y_2 &= -x_1 + x_2
\end{align*}
\]

or

\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

\[ \text{↑ note } \det = +1. \]

But not any linear combination will work. The new basis \((y_1, y_2)\) must span a cell that does contain any points inside or on the boundary, except at the 4 corners (= avg. of one point per cell, since each corner is shared among 4 cells).
The requirement for a valid change of basis is that the $r \times r$ matrix $M$, 

\[
\begin{pmatrix}
  y_1 \\
  \\ \\
  y_r \\
\end{pmatrix} = M 
\begin{pmatrix}
  x_1 \\
  \\ \\
  x_r \\
\end{pmatrix}
\]

must consist of integers and must have an inverse $M^{-1}$ that consists of integers. This means $M \in \text{GL}(r, \mathbb{Z})$. It also means that $\det M = \pm 1$.

Notice if we have a freely generated Abelian group generated by $(x_1, \ldots, x_r)$, then the only way \( \sum_{i=1}^{r} n_i x_i = 0 \) is when $n_i = 0$, for all $i$. Then we may borrow terminology from linear algebra and say that $(x_1, \ldots, x_r)$ are linearly independent.
To handle the general case (free or not free) of a finitely generated Abelian group, let \( G = \) the group, \( \{x_1, \ldots, x_r\} \) a set of generators, and consider the map

\[
 f : \mathbb{Z}^r \rightarrow G : (n_1, \ldots, n_r) \mapsto \sum_{i=1}^{r} n_i x_i.
\]

This map is onto, \( \text{im} f = G \), by the definition of "generators.

The condition that the group is free is precisely the condition \( \ker f = \{(0, \ldots, 0)\} \),

that is, \( \ker f \) is the trivial subgroup of \( \mathbb{Z}^r \) containing the identity.

Then

\[
 G \cong \frac{\mathbb{Z}^r}{\ker f} \cong \mathbb{Z}^r,
\]

same conclusion as above.

But if the group is not free, then \( \ker f \) contains more than the identity element. In fact, it must be a sublattice of \( \mathbb{Z}^r \), since it's closed under addition.

As an example, consider the case \( r=1 \), so only one generator \( x \).

An Abelian group with one generator is called cyclic. If the group is free, then \( G \cong \mathbb{Z} \). But suppose for example, that \( 2x = 0 \).

\[
 \begin{array}{cccccc}
 -2 & -1 & 0 & 1 & 2 & 3 & 4 \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
 \ker f \\
 \end{array}
\]

Then \( \ker f \) is the set \( \{ \ldots, -2, 0, 2, 4, \ldots \} \), the sublattice spanned by \( \{2\} \). Call this subgroup \( 2\mathbb{Z} \).

\( L \) also sublattice.

[More generally, \( k\mathbb{Z} \) is the set \( \{ kn \mid n \in \mathbb{Z} \} \) for \( k \geq 1 \)]

Then

\[
 G = \frac{\mathbb{Z}}{2\mathbb{Z}} \cong \mathbb{Z}_2 = \{0, 1\}, \quad \text{integers modulo } 2.
\]
More generally, \[ \mathbb{Z} / k \mathbb{Z} \cong \mathbb{Z}_k = \{0, 1, \ldots, k-1\} \text{ modulo } k. \]

We see that a cyclic group either contains an \( \infty \) number of elements, in which case it is isomorphic to \( \mathbb{Z} \), or else it contains a finite number \( k \geq 1 \) elements, in which case it is isomorphic to \( \mathbb{Z}_k \).

Now we quote the facts (without proof) for the case of arbitrary \( r \).

As above, \( G \) = finitely generated Abelian group with generators \( \{x_1, \ldots, x_r\} \), and \( f: \mathbb{Z}^r \to G: (n_1, \ldots, n_r) \mapsto \sum_{i=1}^{r} n_i x_i \). Again, \( \text{im} f = G \).

**Fact 1.** Any subgroup of \( \mathbb{Z}^r \) is a sublattice of \( \mathbb{Z}^r \) which can be spanned by some set of integer vectors (elements of \( \mathbb{Z}^r \)), call them \( \{y_1, \ldots, y_p\} \), \( p \leq r \). In particular, \( \ker f \) can be written,

\[
\ker f = \left\{ \sum_{i=1}^{p} m_i y_i \mid m_i \in \mathbb{Z} \right\}.
\]

This is the general form of a finitely generated Abelian group.

**Fact 2.** \[ \frac{\mathbb{Z}^r}{\ker f} \cong \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \ldots \times \mathbb{Z}_{k_p} \times \mathbb{Z} \times \ldots \times \mathbb{Z} \cong G. \]

\( k_i \geq 1, i=1, \ldots, p \).

Note that \( \mathbb{Z}_1 \) (the case \( k=1 \)) is just the trivial group \( \{0\} \) (the cyclic group with one element); if this occurs in the list it can be dropped. Note also that we never had to say that the generators \( \{x_1, \ldots, x_r\} \) were "minimal"; if a smaller set of generators would work, then there will appear \( \mathbb{Z}_1 \) factors in the final product.
Now we turn to the machinery needed to formalize the idea of a polyhedron.

**Def.** A \( r \)-simplex is a region of \( \mathbb{R}^n \) \((n \geq r)\) specified by \( r+1 \) points, \( P_0, P_1, \ldots, P_r \), denoted \( \sigma = \langle P_0 P_1 \ldots P_r \rangle \), and defined by

\[
\sigma = \langle P_0 P_1 \ldots P_r \rangle = \left\{ \sum_{i=0}^{r} c_i P_i \mid \sum_{i=1}^{r} c_i = 1, \ c_i \geq 0 \right\}.
\]

Coefficients \( \{c_i\} \) are barycentric coordinates. The order of the points in \( \langle P_0 \ldots P_r \rangle \) is unimportant (any permutation represents the same simplex.)

**Example:**

\( \sigma_0 = \langle P_0 \rangle = \text{a point} \)

\( \sigma_1 = \langle P_0 P_1 \rangle = \text{an edge} \)

\( \sigma_2 = \langle P_0 P_1 P_2 \rangle = \text{a face (triangle)} \)

**Def.** Let \( \sigma_r = \langle P_0 \ldots P_r \rangle = \text{a simplex} \)

\( \sigma_q = \langle P_{i_0} \ldots P_{i_q} \rangle = \text{a q-face of } \sigma_r, \quad (q \leq r) \)

where \( \{P_{i_0}, \ldots, P_{i_q}\} \) is a subset of \( \{P_0, \ldots, P_r\} \).

Denote \( \sigma_q \leq \sigma_r \).

**Example:**

\[
\# q\text{-faces} = \binom{r+1}{q+1}
\]

\( \sigma_3 \text{ tetrahedron} = 3\text{-simplex} \)

\[\begin{align*}
4 & \text{ 2-faces} \\
6 & \text{ 1-faces} \\
4 & \text{ 0-faces}
\end{align*}\]
Def. A simplicial complex is a set $K$ of simplexes such that:
1) if $\sigma \in K$ then $K$ also contains all the faces of $\sigma$.
2) if $\sigma, \sigma' \in K$ then either $\sigma \cap \sigma' = \emptyset$ or $\sigma$ and $\sigma'$ intersect in a common face.

Property 2 means that simplicial complexes are "nicely fitted together."

Def. A polyhedron $|K|$ is the set

$$|K| = \bigcup_{\sigma \in K} \sigma$$

Def. A topological space is triangulable if $\exists$ a polyhedron $|K|$ and homeomorphism $f: |K| \to X$.

Example: A cylinder (square with 2 sides identified, $\square \square$).

This

$$\begin{array}{c}
\text{doesn't work, because } |K| \text{ is not homeomorphic to the cylinder (it is two rectangles superimposed, i.e., one rectangle).}
\end{array}$$

Instead, use

$$\begin{array}{c}
\text{6 triangles}
\end{array}$$

Note: the triangulation is not unique, because we can always refine it (add more simplexes).
Now oriented simplices. Look at e.g. 1-simplex.

\[
\langle p_0 p_1 \rangle = \left\langle \begin{array}{c} p_1 \\ p_0 \end{array} \right\rangle = \langle p_1 p_0 \rangle \quad \text{unordered.}
\]

Change of notation, write \((p_0 p_1)\) for ordered simplex,

\[
(p_0 p_1) = \left\langle \begin{array}{c} p_1 \\ p_0 \end{array} \right\rangle = -(p_1 p_0).
\]

For 2-simplexes, have 3 points, \((p_0 p_1 p_2)\). Declare that this changes sign if points subjected to an odd permutation. Generally,

\[
(p_{i_0} p_{i_1} \ldots p_{i_n}) = \pm (p_{i_0} \ldots p_{i_n})
\]

where \(\pm\) = sign of permutation \((0 \ 1 \ \ldots \ n)\).

Special case of 0-dimension. \(\langle p_0 \rangle = * \quad \text{a point.}\)

\(\langle p_0 \rangle = "\text{oriented}" \text{ point.} \ = \ =\)

As discussed previously, motivation for oriented simplices is use in integrals, e.g., Stokes’ thm. Here is what we mean by a “0-dimensional integral”:

\[
\int f = f(p_0)
\]

where \(p_0 \in M \quad f : M \to \mathbb{R} \quad \text{(any)}\).

Thus,

\[
\int f = -f(p_0), \text{ etc.}
\]

Thus linear combinations of oriented simplices become meaningful (as things you integrate over).
Above we defined a simplicial complex $K$ as a collection of unoriented simplices. Now we modify the definition in obvious ways to talk about an oriented simplicial complex $K$: it is a set of oriented simplices $\{\sigma_i\}$. (Use the same symbols)

Let $K$ = an oriented simplicial complex $= \{\sigma_i\}$. Define:

**Def.** The $r$-th chain group $C_r(K)$ is the free Abelian group generated by the $r$-dimensional, oriented simplices in $K$, that is, it is the set of formal linear combinations,

$$c = \sum_i n_i \sigma_i, \quad n_i \in \mathbb{Z}, \quad c \in C_r(K).$$

where $\{\sigma_i\}$ is the set of $r$-simplices in $K$, whose number is $I_r$.

$c$ is called an $r$-chain. Note, $C_r(K) \cong \mathbb{Z}^{I_r}$.

Now we motivate the definition of the boundary operator. Start with an oriented 2-simplex $(p_0, p_1, p_2)$

\[ \triangle \]

The arrow is a convenient way of specifying this orientation; since any cyclic permutation of $(p_0, p_1, p_2)$ is an even permutation. (The opposite direction would reverse the sign.) We define the boundary operator in this example by writing

\[ \partial \triangle = \partial \]

or

\[ \partial (p_0 p_1 p_2) = (p_0 p_1) + (p_1 p_2) + (p_2 p_0). \]
Notice that $\partial$ acting on a 2-simplex is not a simplex, but rather a linear combination of simplexes (a chain). Another example, on the boundary of a 1-simplex:

$$\partial (p_0 p_1) = (p_1) - (p_0)$$

$$\partial \int_{p_0}^{p_1} p_1 + p_0$$

In general, we define $\partial_r: C_r(K) \to C_{r-1}(K)$ by giving its action on an $r$-simplex and then extending in the obvious way to linear combinations. If $\sigma_r = (p_0, \ldots, p_r)$ is an oriented $r$-simplex, then we define

$$\partial_r \sigma_r = \sum_{i=0}^{r} (-1)^i (p_0 \ldots \hat{p}_i \ldots p_r)$$

example:

$$\partial_2 (p_0 p_1 p_2) = (p_1 p_2) - (p_0 p_2) + (p_0 p_1),$$

same answer as before since $-(p_0 p_2) = +(p_2 p_0)$.

One more note about the boundary operator. In special case $r=0$, we define

$$\partial_0 (p_0) = 0,$$

since there are no $(-1)$-chains.

Also note, $\partial_r: C_r(K) \to C_{r-1}(K)$ is a group homomorphism (it commutes with $+$).
There is really one body operator for each dimension. Each maps \( \partial_r \) of \( C_r(k) \) into \( C_{r-1}(k) \). So we have a sequence of maps, \( (n=dim(k)) \)

\[
C_n(k) \xrightarrow{\partial_n} C_{n-1}(k) \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} C_1(k) \xrightarrow{\partial_1} C_0(k) \xrightarrow{\partial_0} \{0\}.
\]

**Def:** An chain \( c \in C_r(k) \) such that \( \partial c = 0 \) is called a cycle (or \( r \)-cycle). A cycle is a chain without a boundary.

**Example:**

\[
\begin{array}{c}
\text{P}_2 \\
\text{P}_0 \\
\text{P}_1
\end{array}
\]

\[
c = (p_0p_1) + (p_1p_2) + (p_2p_0)
\]

\[
\partial c = (p_1) - (p_0) + (p_2) - (p_1) + (p_0) - (p_2) = 0.
\]

\( c \) is a cycle (1-cycle).

**Another example:** Any 0-chain is a cycle, since \( \partial (p_0) = 0 \).

**Def:** The set

\[
Z_r(k) = \{ c \in C_r(k) | \partial c = 0 \} = \text{ker} \partial_r
\]

is the \( r \)-th cycle group. It is obviously a subgroup of \( C_r(k) \), which means (see notes with HW6) that it is a sublattice of \( C_r(k) \) and that it (like \( C_r(k) \)) is a free, finitely generated Abelian group.

**Special case:** \( Z_0(k) = C_0(k) \) (all 0-chains are cycles).
Def: The set

\[ B_r(k) = \{ b \in C_r(k) \mid b = \partial_{r+1} c, \text{ some } c \in C_{r+1}(k) \} = \text{im} \ \partial_{r+1} \]

is the r-th boundary group. Elements of \( B_r(k) \) are called r-boundaries. \( B_r(k) \) is obviously a subgroup of \( C_r(k) \), hence it is a free, finitely generated Abelian group.

For special case: \( r = n \), since there are no \((n+1)\)-simplexes, we define \( B_n(k) = \text{"im} \ \partial_{n+1} \text{"} = \{0\}\).
1st term = $\sum_{\nu < \xi} (-1)^{\nu + \xi} (p_{\nu} \cdots \hat{p}_{\xi} \cdots p_{\nu + 1})$

2nd term = $-\sum_{\nu < \xi} (-1)^{\nu + \xi} (p_{\nu} \cdots \hat{p}_{\xi} \cdots p_{\nu + 1}) = -1$st term by swapping $i$,$j$.

QED.

Immediate corollary: $B_{r}(k) \subseteq Z_{r}(k)$.

If $b \in B_{r}(k)$, then $b = \partial c$, some $c \in C_{r+1}(k)$.

Hence $\exists b = \partial c = 0$ hence $b \in Z_{r}(k)$.

Altogether, $B_{r}(k) \subseteq Z_{r}(k) \subseteq C_{r}(k)$.

Actually $\subseteq$ means "subgroup". All 3 groups are free, finitely generated Abelian group.

Finally, we define $H_{r}(K) = \frac{Z_{r}(k)}{B_{r}(k)} = r$-th homology group.

$H_{r}(K)$ is a topological invariant, that is, if $1K$ and $1K'$ are homeomorphic, then $H_{r}(K) = H_{r}(K')$. If you have a topological space homeomorphic to $1K$, then $H_{r}(K)$ is regarded as the same as $H_{r}(K)$.

Note that if $h \in H_{r}(K)$, then $h$ is an equivalence class of cycles whose difference is a boundary, $h = [c]$, $c \in Z_{r}(k)$, $c - c' \in B_{r}(k)$.