Examples of Abelian groups include:

$$\mathbb{R}^n,\ S^1,\ \mathbb{Z},\ \mathbb{Z}_k$$

vector space $\quad \{e^{i\theta}\}. \quad$ integers modulo k $= \{0, 1, \ldots, k-1\}$.

We will only be interested in <u>discrete</u> Abelian groups, which excludes things like $\mathbb{R}^n$ and S^1.

If G is ~~an~~ a discrete Abelian group and $x_1, \ldots, x_r$ are elements of G such that any $g \in G$ can be written in the form,

$$g = \sum_{i=1}^{r} n_i x_i, \qquad n_i \in \mathbb{Z},$$

then G is said to be <u>generated</u> by the $\{x_i\}$ and the $\{x_i\}$ are said to be the <u>generators</u>. If $r < \infty$, then we say that G is <u>finitely generated</u>. For homology theory we only need finitely generated Abelian groups. Notice that so far we're not saying that the generators are minimal in number (and in any case they are not unique).

Either the group elements $\sum_{i=1}^{r} n_i x_i$ for $n_i \in \mathbb{Z}$ are all unique, or there is some duplication (some group elements can be represented as a "linear combination" of the generators in more than one way).

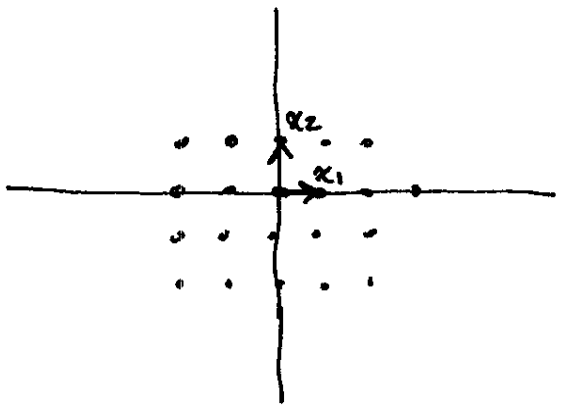
In the former case we say that the group is <u>freely generated</u>, and that G is a <u>free Abelian group of rank r</u>. In this case, every $g \in G$ can be <u>uniquely</u> represented as

$$g = \sum_{i=1}^{r} n_i x_i,$$

and G is isomorphic to $\mathbb{Z}^r$, $\quad G \cong \mathbb{Z}^r, \quad g \mapsto (n_1, \ldots, n_r)$.

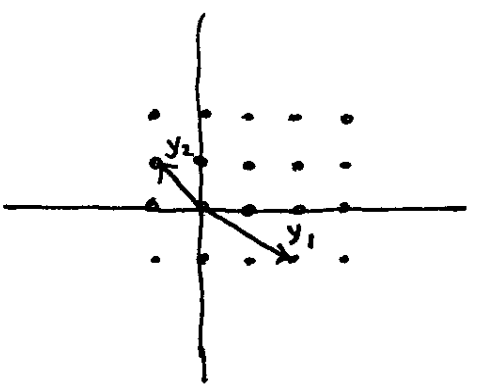
In effect, there <u>is</u> only one free Abelian group of rank r, it is $\mathbb{Z}^r$, and it can be visualized as the integer lattice in r-dimensional $\mathbb{R}^r$.

About the non-uniqueness of the generators. Take the case $r=2$ for illustration. Represent the generators x_1, x_2 as unit vectors in the plane, and the group as the lattice.



The "basis" (x_1, x_2) spans the lattice, but it is not unique. We could use

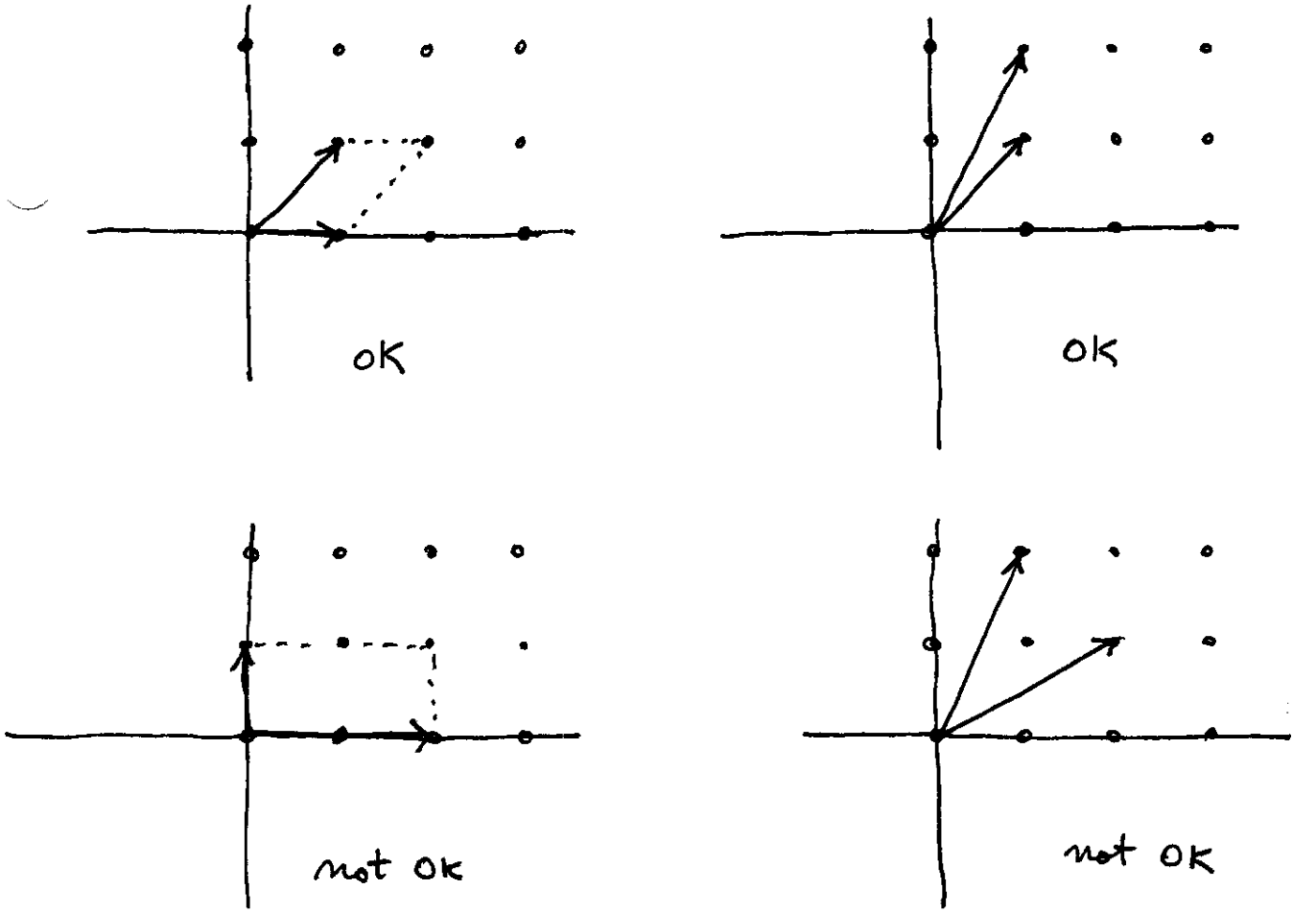
$$y_1 = 2x_1 - x_2$$
$$y_2 = -x_1 + x_2$$



or $$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

↑ note det = +1.

But not any linear combination will work. The new basis (y_1, y_2) must span a cell that does contain any points inside or on the boundary, except at the 4 corners (= avg. of one point per cell, since each corner is shared among 4 cells).



The requirement for a valid change of basis is that the $r \times r$ matrix M,

$$\begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} = M \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix}$$

must consist of integers and must have an inverse M^{-1} that consists of integers. This means $M \in GL(r, \mathbb{Z})$. It also means that $\det M = \pm 1$.

Notice if we have a freely generated Abelian group of rank generated by $(x_1, \ldots, x_r)$, then the only way $\sum_{i=1}^{r} n_i x_i = 0$ is when $n_i = 0$, $\forall i$. Then we may borrow terminology from linear algebra and say that $(x_1, \ldots, x_r)$ are <u>linearly</u> <u>independent</u>.

To handle the general case (free or not free) of a finitely generated Abelian group, let G = the group, $\{x_1, \dots, x_r\}$ a set of generators, and consider the map

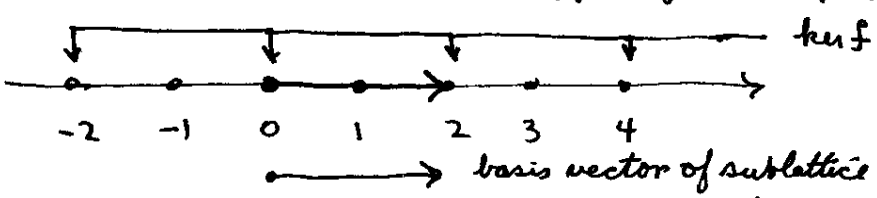
$$f: \mathbb{Z}^r \to G: \ (n_1, \dots, n_r) \mapsto \sum_{i=1}^{r} n_i x_i .$$

This map is <u>onto</u>, $\operatorname{im} f = G$, by the definition of "generators".
The condition that the group is free is precisely the condition $\ker f = \{(0, \dots, 0)\}$, that is, $\ker f$ is the trivial subgroup of $\mathbb{Z}^r$ containing the identity.
Then

$$G \cong \frac{\mathbb{Z}^r}{\ker f} \cong \mathbb{Z}^r, \quad \text{same conclusion as above.}$$

But if the group is not free, then $\ker f$ contains more than the identity element. In fact, it must be a <u>sublattice</u> of $\mathbb{Z}^r$, since it's closed under addition.

As an example, consider the case $r = 1$, so only one generator x. An Abelian group G with one generator is called <u>cyclic</u>. If the group is free, then $G \cong \mathbb{Z}$. But suppose for example, that $2x = 0$.



Then $\ker f$ is the set $(\dots, -2, 0, 2, 4, \dots)$, the sublattice spanned by (2). Call this subgroup $2\mathbb{Z}$.
↳ also sublattice.

[More generally, $k\mathbb{Z}$ is the set $\{kn \mid n \in \mathbb{Z}\}$ for $k \geq 1$]

Then $G = \dfrac{\mathbb{Z}}{2\mathbb{Z}} \cong \mathbb{Z}_2 = \{0, 1\}$, integers modulo 2.

More generally, $\dfrac{\mathbb{Z}}{k\mathbb{Z}} \cong \mathbb{Z}_k = \{0, 1, \ldots, k-1\}$ = integers modulo k.

We see that a cyclic group either contains an ∞ number of elements, in which case it is isomorphic to $\mathbb{Z}$, or else it contains a finite number $k \geq 1$ elements, in which case it is isomorphic to $\mathbb{Z}_k$.

Now we quote the facts (without proof) for the case of arbitrary r.

As above, G = finitely generated Abelian group with generators $\{x_1, \ldots, x_r\}$, and $f: \mathbb{Z}^r \to G : (n_1, \ldots, n_r) \mapsto \sum_{i=1}^{r} n_i x_i$. Again, $\operatorname{im} f = G$.

<u>Fact 1.</u> Any subgroup of $\mathbb{Z}^r$ is a sublattice of $\mathbb{Z}^r$ which can be spanned by some set of integer vectors (elements of $\mathbb{Z}^r$), call them (lin. independent.) $\{y_1, \ldots, y_p\}$, $p \leq r$. In particular, $\ker f$ can be written,

$$\ker f = \left\{ \sum_{i=1}^{p} m_i y_i \;\middle|\; m_i \in \mathbb{Z} \right\}.$$

This is the general form of a finitely generated Abelian group.

<u>Fact 2.</u>

$$\frac{\mathbb{Z}^r}{\ker f} \cong \underbrace{\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \ldots \times \mathbb{Z}_{k_p}}_{p \text{ factors}} \times \underbrace{\mathbb{Z} \times \ldots \times \mathbb{Z}}_{r-p \text{ factors.}} \cong G.$$

$k_i \geq 1$, $i = 1, \ldots, p$.

Note that $\mathbb{Z}_1$ (the case $k=1$) is just the trivial group $\{0\}$ (the cyclic group with one element); if this occurs in the list it can be dropped. Note also that we never had to say that the generators $\{x_1, \ldots, x_r\}$ were "minimal"; if a smaller set of generators would work, then there will appear $\mathbb{Z}_1$ factors in the final product.

Now we turn to the machinery needed to formalize the idea of a polyhedron.

Def. A <u>r-simplex</u> is a region of $\mathbb{R}^m$ $(m \geq r)$ specified by $r+1$ "geometrically independent" points, $p_0, p_1, \ldots, p_r$, denoted $\sigma = \langle p_0 p_1 \cdots p_r \rangle$, and defined by

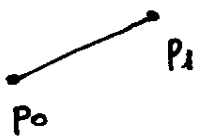
$$\sigma = \langle p_0 p_1 \cdots p_r \rangle = \left\{ \sum_{i=0}^{r} c_i p_i \;\middle|\; \sum_{i=1}^{r} c_i = 1, \quad c_i \geq 0 \right\}.$$

Coefficients $\{c_i\}$ are <u>barycentric coordinates</u>. The <u>order</u> of the points in $\langle p_0 \cdots p_r \rangle$ is unimportant (any permutation represents the same simplex.)

<u>Examples:</u>

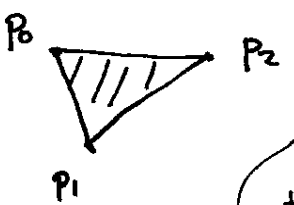
$\sigma_0 = \langle p_0 \rangle$ = a point

$\sigma_1 = \langle p_0 p_1 \rangle$ = an edge



$x = c_0 p_0 + c_1 p_1$
$= c_0 p_0 + (1-c_0) p_1$
$0 \leq c_0 \leq 1.$

$\sigma_2 = \langle p_0 p_1 p_2 \rangle$ = a face (triangle)



Here, "geometrically independent" means $p_0 p_1 p_2$ do not lie on a line: the triangle has positive area.

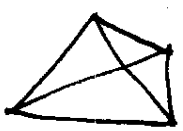
Def. Let $\sigma_r = \langle p_0 \cdots p_r \rangle$ = a simplex

$\sigma_q = \langle p_{i_0} \cdots p_{i_q} \rangle$ = a q-face of σ_r, $(q \leq r)$

where $\{p_{i_0}, \cdots p_{i_q}\}$ is a subset of $\{p_0, \ldots, p_r\}$.

Denote $\sigma_q \leq \sigma_r$.

<u>Example:</u>



σ_3 = tetrahedron = 3-simplex

4 2-faces
6 1-faces
4 0-faces

$$\#\ q\text{-faces} = \binom{r+1}{q+1}$$

Def. A <u>simplicial complex</u> is a set $K = \{\sigma_i\}$ of simplexes such that:

1) If $\sigma \in K$ then K also contains all the faces of σ
2) If $\sigma, \sigma' \in K$ then either $\sigma \cap \sigma' = \phi$ or σ and σ' intersect in a common face.

Property 2 means that simplicial complexes are "nicely fitted together"

ok ok ok not ok

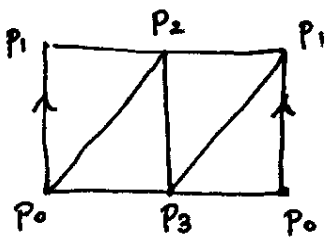
Def. A <u>polyhedron</u> $|K|$ is the set

$$|K| = \bigcup_{\sigma_i \in K} \sigma_i .$$

Def. A topological space X is <u>triangulable</u> if $\exists$ a polyhedron $|K|$ and homeomorphism $f: |K| \to X$.

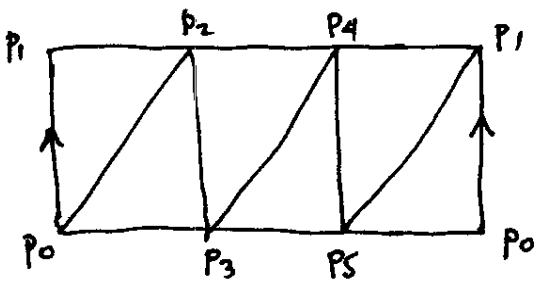
<u>Example:</u> A cylinder (square with 2 sides identified, ).

This



doesn't work, because $|K|$ is not homeomorphic to the cylinder (it is two rectangles superimposed, i.e., one rectangle).

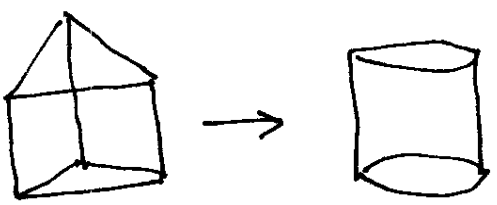
Instead, use



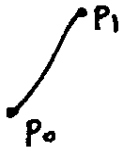
6 triangles.

<u>Note:</u> the triangulation is <u>not</u> <u>unique</u>, because we can always <u>refine</u> it (add more simplices).

$f:$

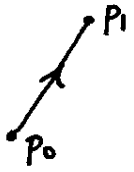


Now oriented simplexes. Look at e.g. 1-simplex.

$\langle p_0 p_1 \rangle =$ 

$= \langle p_1 p_0 \rangle$ unordered.

Change of notation, write $(p_0 p_1)$ for ordered simplex,

$(p_0 p_1) =$ 

$= -(p_1 p_0)$.

For 2-simplexes, have 3 points, $(p_0 p_1 p_2)$. Declare that this changes sign if points subjected to an odd permutation. Generally,

$$(p_{i_0} p_{i_1} \cdots p_{i_n}) = \pm (p_0, \ldots, p_n)$$

where $\pm$ = sign of permutation $\begin{pmatrix} 0 & 1 & \ldots & n \\ i_0 & i_1 & \ldots & i_n \end{pmatrix}$.

Special case of 0-dimension. $\langle p_0 \rangle = \bullet$ a point.

(p_0) = "oriented" point.

As discussed previously, motivation for oriented simplexes is use in integrals, e.g., Stokes' thm. Here is what we mean by a "0-dimensional integral":

$$\int_{(p_0)} f = f(p_0)$$

where $p_0 \in M$
$f : M \to \mathbb{R}$ (say).

Thus,

$$\int_{-(p_0)} f = -f(p_0), \text{ etc.}$$

Thus linear combinations of oriented simplexes become meaningful (as things you integrate over).

Above we defined a simplicial complex K as a collection of unoriented simplexes. Now we modify the definition in obvious ways to talk about an <u>oriented simplicial complex</u> K: it is a set of <u>oriented simplexes</u> $\{\sigma_i\}$. (Use the same symbols.)

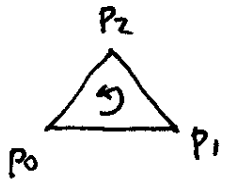
Let K = an oriented simplicial complex = $\{\sigma_i\}$. Define:

<u>Def</u>: The <u>r-th chain group</u> $C_r(K)$ is the free Abelian group generated by the r-dimensional, oriented simplexes in K, that is, it is the set of formal linear combinations,

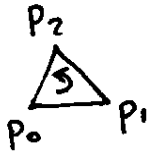
$$c = \sum_i^{I_r} n_i \, \sigma_{ri}, \qquad n_i \in \mathbb{Z}, \qquad c \in C_r(K).$$

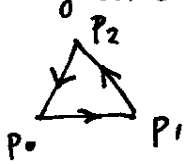
where $\{\sigma_{ri}\}$ is the set of r-simplexes in K, whose number is I_r. c is called an <u>r-chain</u>. Note, $C_r(K) \cong \mathbb{Z}^{I_r}$.

Now we motivate the definition of the <u>boundary operator</u>. Start with an oriented 2-simplex $(p_0 p_1 p_2)$



The arrow is a convenient way of specifying this orientation, since any cyclic permutation of $(p_0 p_1 p_2)$ is an even permutation. (The opposite direction would reverse the sign.) We define the boundary operator in this example by writing

∂ 

$=$ 

or

$$\partial (p_0 p_1 p_2) = (p_0 p_1) + (p_1 p_2) + (p_2 p_0).$$

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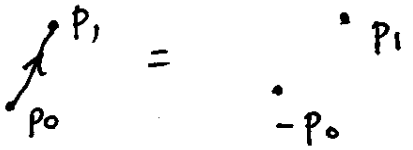
Notice that ∂ acting on a 2-simplex is not a simplex, but rather a linear combination of simplexes (a chain). Another example, the boundary of a 1-simplex:

$$\partial\,(p_0 p_1) = (p_1) - (p_0)$$

∂ 

In general, we define $\partial_r : C_r(K) \to C_{r-1}(K)$ by giving its action on an r-simplex and then extending in the obvious way to linear combinations. If $\sigma_r = (p_0, \ldots, p_r)$ is an oriented r-simplex, then we define

hat means omit this point.

$$\partial_r \sigma_r = \sum_{i=0}^{r} (-1)^i \, (p_0 \ldots \hat{p}_i \ldots p_r)$$

E.g., $\partial_2\,(p_0 p_1 p_2) = (p_1 p_2) - (p_0 p_2) + (p_0 p_1)$, same answer as before since $-(p_0 p_2) = +(p_2 p_0)$.

One more note about defn of bdry operator. In special case $r=0$, we define

$$\partial_0\,(p_0) = 0,$$

since there are no (-1)-chains.

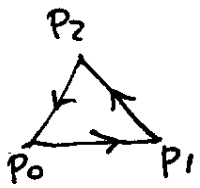
Also note, $\partial_r : C_r(K) \to C_{r-1}(K)$ is a group homomorphism (it commutes with $+$).

There is really one bdry operator for each dimension. Each maps $C_r(K)$ into $C_{r-1}(K)$. So we have a sequence of maps, ($n = \dim K$) 9/9/08

$$C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} \{0\}.$$

<u>Def</u>: An r-chain $c \in C_r(K)$ such that $\partial c = 0$ is called a <u>cycle</u> (or <u>r-cycle</u>). A cycle is a chain without a boundary.

<u>Example</u>:



$$c = (p_0 p_1) + (p_1 p_2) + (p_2 p_0)$$

$$\partial c = (p_1) - (p_0) + (p_2) - (p_1) + (p_0) - (p_2) = 0.$$

c is a cycle (1-cycle).

<u>Another example</u>: Any 0-chain is a cycle, since $\partial(p_0) = 0$.

<u>Def</u>: The set

$$Z_r(K) = \{ c \in C_r(K) \mid \partial_r c = 0 \} = \ker \partial_r$$

is the <u>r-th cycle group</u>. It is obviously a subgroup of $C_r(K)$, which means (see notes with HW 3) that it is a sublattice of $C_r(K)$ and that it (like $C_r(K)$) is a free, finitely generated Abelian group.

Special case: $Z_0(K) = C_0(K)$ (all 0-chains are cycles).
($r=0$)

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Def: The set

$$B_r(K) = \{ b \in C_r(K) \mid b = \partial_{r+1} c, \text{ some } c \in C_{r+1}(K) \} = \operatorname{im} \partial_{r+1}$$

is the <u>r-th boundary group</u>. Elements of $B_r(K)$ are called <u>r-boundaries</u>. $B_r(K)$ is obviously a subgroup of $C_r(K)$, hence it is a free, finitely generated Abelian group.

<u>Special case</u>: For $r = n$, since there are no $(n+1)$-simplexes, we define $B_n(K) = \text{"}\operatorname{im} \partial_{n+1}\text{"} = \{0\}$.

Thm: $\partial_r \partial_{r+1} = 0$. The boundary of a boundary vanishes.

<u>Proof</u>: It suffices to consider an $(r+1)$-simplex (a basis vector in $C_{r+1}(K)$), $(p_0, \dots, p_{r+1})$:

$$\partial_{r+1}(p_0, \dots, p_{r+1}) = \sum_{i=0}^{r+1} (-1)^i (p_0 \dots \hat{p}_i \dots p_{r+1})$$

$$\partial_r \partial_{r+1}(p_0, \dots, p_{r+1}) = \sum_{i=0}^{r+1} (-1)^i \Bigg[\sum_{j=0}^{i-1} (-1)^j (p_0 \dots \hat{p}_j \dots \hat{p}_i \dots p_{r+1})$$

$$+ \sum_{j=i+1}^{r+1} (-1)^{j+1} (p_0 \dots \hat{p}_i \dots \hat{p}_j \dots p_{r+1}) \Bigg] = 2 \text{ terms}$$

~~Swap i, j in 2nd term, it becomes~~

$$\cancel{- \sum_{i=0}^{r+1} (-1)^i \sum_{j=}}$$

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$$\text{1st term} = \sum_{i>j} (-1)^{i+j} \left(p_0 \dots \hat{p}_j \dots \hat{p}_i \dots p_{r+1}\right)$$

$$\text{2nd term} = -\sum_{i<j} (-1)^{i+j} \left(p_0 \dots \hat{p}_i \dots \hat{p}_j \dots p_{r+1}\right) = -\text{1st term by swapping } i, j.$$

QED.

Immediate <u>corollary</u>: $B_r(K) \subseteq Z_r(K)$.

If $b \in B_r(K)$, then $b = \partial c$, some $c \in C_{r+1}(K)$.

Hence $\partial b = \partial\partial c = 0$ hence $b \in Z_r(K)$.

Altogether, $\boxed{B_r(K) \subseteq Z_r(K) \subseteq C_r(K)}$

Actually $\subseteq$ means "subgroup". All 3 groups are free, finitely generated Abelian group.

Finally, we define

Note, for $r > \dim K$, $H_r(K)$ is understood to be $\{0\}$ (the trivial group).

$$H_r(K) = \frac{Z_r(K)}{B_r(K)} = r\text{-th homology group.}$$

$H_r(K)$ is a topological invariant, that is, if $|K|$ and $|K'|$ are homeomorphic, then $H_r(K) = H_r(K')$. If you have a topological space X homeomorphic to $|K|$, then $H_r(X)$ is regarded as the same as $H_r(K)$.

Note, if $h \in H_r(K)$, then h is an equivalence class of cycles whose difference is a boundary,

$$h = [c], \quad c \in Z_r(K),$$
$$c \sim c' \text{ if } c - c' \in B_r(K).$$