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Let's look at some of the structure that exists at the level of a topological space.

Let $X =$ a topological space.

Def. A subset $A \subset X$ is closed if the complement $X - A$ is open.

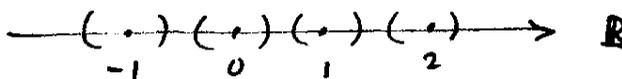
Also define closure, interior (see text).

Definition of continuity given above.

Now compactness, a property that is a little mysterious if you've never taken a course in topology.

Def. A subset $A \subset X$ of a topological space is compact if every open cover contains a finite subcover. (An open cover is a set of open sets whose union contains A .)

Example, $\mathbb{Z} \subset \mathbb{R}$.



$$X = \mathbb{R}$$

$$A = \mathbb{Z}$$

Open cover has ∞ # of open intervals, surrounding each integer. Remove any one and \mathbb{Z} is no longer covered. Therefore \mathbb{Z} is not compact in \mathbb{R} .

Important thm. for compactness in \mathbb{R}^n (Heine-Borel):

Thm. A subset of \mathbb{R}^n is compact if and only if it is closed and bounded.

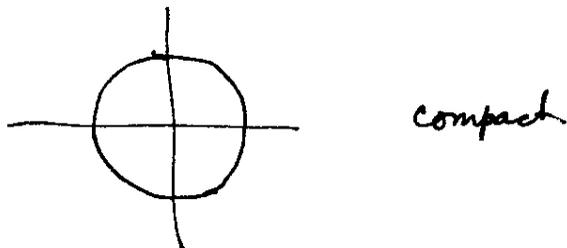
Examples: 1) Open interval in \mathbb{R}



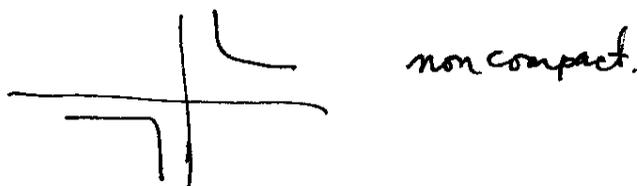
2) Closed " " "



3) S^1 (circle) in \mathbb{R}^2 :



4) Hyperbola in \mathbb{R}^2



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Mention compactness has on group representation theory, e.g.

compact groups such as $SU(n)$, $SO(n)$, etc.

vs. noncompact ones like Lorentz, $GL(n, \mathbb{R})$, $Sp(2n)$ etc.

Now, Connectedness. There are 2 kinds of notions, connected and arc-wise connected, which are the same in most physical applications. Idea corresponds to intuitive notion of connectedness. Official defns:

A topological space X is connected if it cannot be written as the disjoint union of two open sets.

It is arc-wise connected if for $\forall x, y \in X$, there exists a continuous curve in X connecting x and y (i.e., a continuous map: $[0, 1] \rightarrow X$ with $f(0) = x$ and $f(1) = y$.)

Finally, X is simply connected if every closed loop can be continuously contracted to a point.

Now we turn to the principal notion of topological equivalence, namely, the homeomorphism.

Def. Given two topological spaces X and Y . A map $f: X \rightarrow Y$ is said to be a homeomorphism if it is continuous and possesses a continuous inverse $f^{-1}: Y \rightarrow X$. ~~has an~~ If such an f exists, X and Y are said to be homeomorphic.

Nakahara discusses this concept in the framework of a continuous deformation of one space into another (the coffee mug to the doughnut, for example). Continuous deformation, however, requires an embedding space (in which X and Y are subsets), and this plays no part in the definition of homeomorphism.

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topological

A central problem of topology is to classify spaces, to within a homeomorphism. One can easily show that the relation "homeomorphic" is an equivalence relation, thus topological spaces are divided into equivalence classes.

It is not known how to classify all equivalence classes of topological spaces. That is, given two spaces, it may not be easy to show that they are homeomorphic, apart from finding the homeomorphism that connects them. However, it may be relatively easy to show that they are not homeomorphic, by using topological invariants.

A topological invariant is a quantity or characteristic that is invariant under homeomorphisms. Thus, if two spaces have different invariants, they are not homeomorphic.

see book for more on topological invariants, Euler characteristic.

Examples of topological invariants:

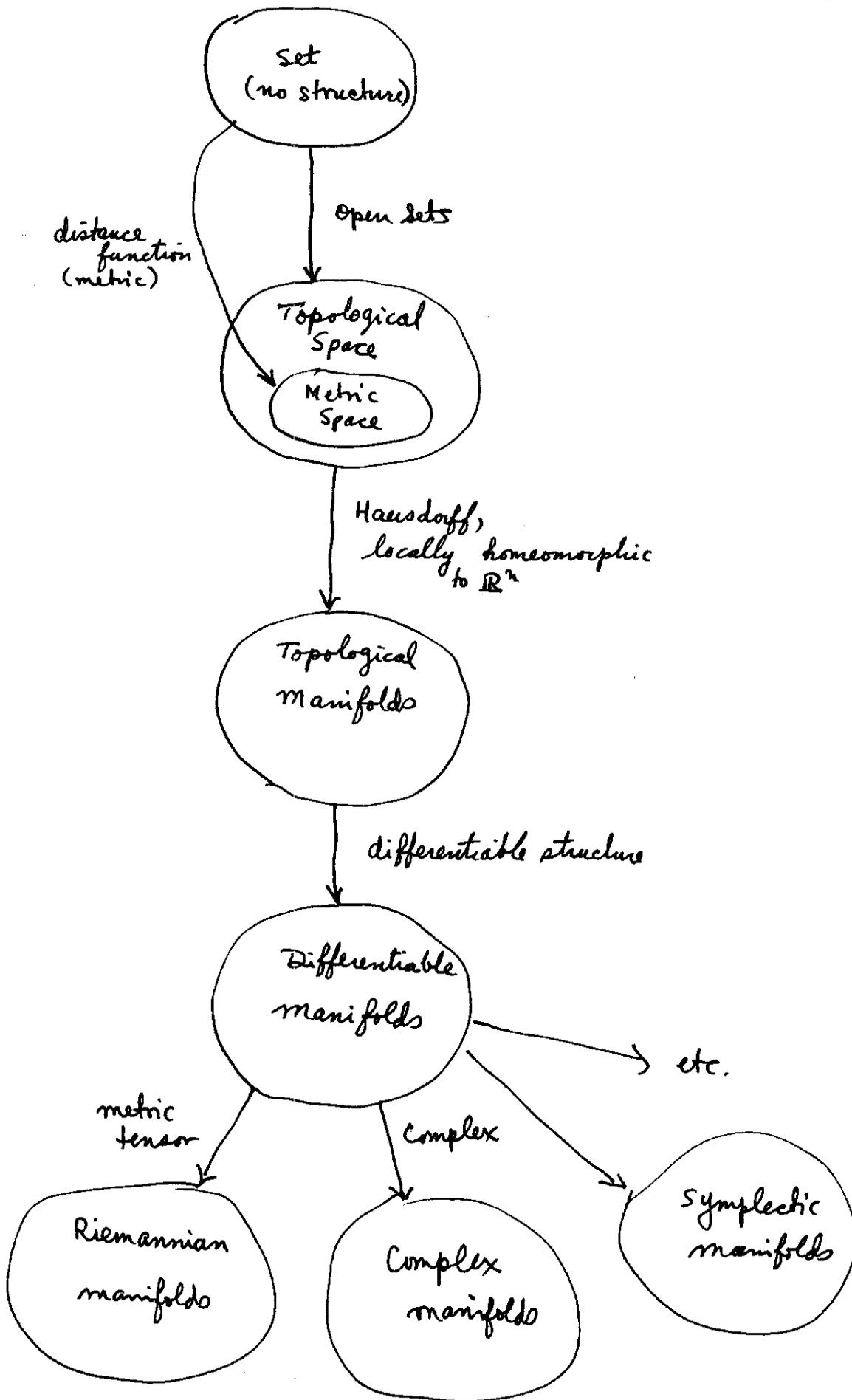
1. Compactness
2. Connectedness
3. Simply-Connectedness

(and many more)

Now we restrict topological spaces further by requiring the Hausdorff property, and requiring that the space be "locally homeomorphic to \mathbb{R}^n ". The result is a topological manifold. We will deal exclusively with topological manifolds. At this level we can talk about continuity, but not differentiability. The latter requires additional structure (later in course) to turn a topological manifold into a differentiable manifold.

Finally, a differentiable manifold may be given additional structure (such as a metric tensor) to get a Riemannian manifold, complex manifold, symplectic manifold, etc.

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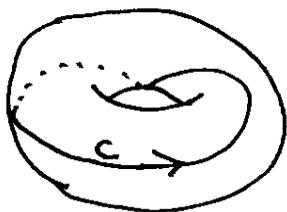


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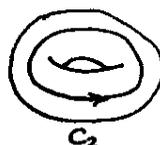
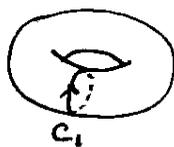
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On \mathbb{R}^3 , every 1-cycle is a boundary, $C_1 = \partial A_1$, $C_2 = \partial A_2$, so their difference is a boundary, too, $C_1 - C_2 = \partial(A_1 - A_2)$, and all 1-cycles belong to the same equivalence class. On \mathbb{R}^3 , there is only one equivalence class. $[C]$, any 1-cycle C , in particular, $C = 0$ (the curve that doesn't go anywhere).

But on the 2-torus, two 1-cycles are equivalent iff their "winding numbers" n_1, n_2 are the same.



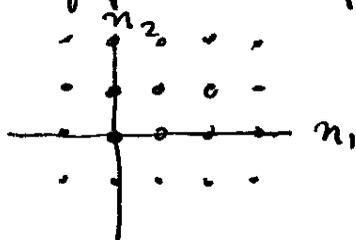
C has winding numbers $(1, 1)$.



$C = C_1 + C_2$.

Thus the equivalence classes of 1-cycles on T^2 are numbered by 2 integers (n_1, n_2) . The space of equivalence classes is $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$.

Can think of this as lattice of points in a plane:



Can also think of it as an Abelian group, with composition law

$$(n_1, n_2) (m_1, m_2) = (n_1 + m_1, n_2 + m_2).$$

(vector addition of lattice vectors). This is expressed by saying,

$$H_1(T^2) = \mathbb{Z}^2$$

$$H_1(\mathbb{R}^3) = \{0\}.$$

$H_1(\text{manifold}) =$ 1st homology group of the manifold.

It is a topological invariant.

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What does it mean to use expressions like $-C$, $C_1 + C_2$, $C_1 - C_2$, etc?

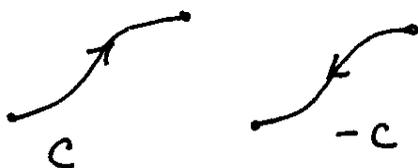
Let $C =$ any ~~closed~~ oriented curve on a manifold M (not necessarily a cycle).



Formally, this is a map $C: [0, 1] \rightarrow M: t \mapsto x(t)$. It is something you integrate over,

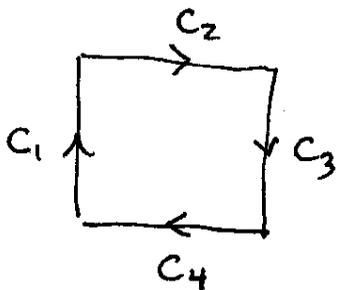
$$\int_C \alpha$$

where $\alpha =$ a differential 1-form (like $\vec{F} \cdot d\vec{x}$) on \mathbb{R}^3 . Let $-C$ be the same curve traversed in the opposite direction,



So that
$$\int_{-C} \alpha = - \int_C \alpha.$$

If you have several segments of a curve, (not necessarily concatenated) define their sum by sums of integrals:



$$\int_{C_1 + C_2 + C_3 + C_4} \alpha = \int_{C_1} \alpha + \int_{C_2} \alpha + \int_{C_3} \alpha + \int_{C_4} \alpha.$$

Thus we can define "linear combinations" of curves with integer coefficients, and

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$$\int_{\sum_i n_i C_i} \alpha = \sum_i n_i \int_{C_i} \alpha, \quad n_i \in \mathbb{Z}.$$

These linear combinations of 1-dimensional, oriented curves with integer coefficients are called 1-chains. They are things you integrate 1-forms over.

Thus a 1-chain is a formal linear combination of oriented 1-curves with integer coefficients.

[Actually, in homotopy theory, one can choose the coefficients to be things other than integers. The favorite choices are \mathbb{Z} , \mathbb{R} and \mathbb{Z}_2 (the spaces from which the coefficients are chosen). For now we use only \mathbb{Z} , but later we'll return to other types of coefficients.]

There is an ^{huge} infinite number of possible, ^{oriented} curves on a given manifold. In our approach to ~~homotopy~~ ^{homology} theory, we will reduce this to a finite number by using a triangulation of a manifold (this is a homeomorphism between a polyhedron in \mathbb{R}^n and the manifold M we wish to study).

The set of 1-chains that can be formed out of a finite # of ^{N distinct} oriented curves $\{C_1, \dots, C_N\}$ is

$$\sum_{i=1}^N n_i C_i, \quad n_i \in \mathbb{Z},$$

it is the space $\mathbb{Z}^N = \mathbb{Z} \times \dots \times \mathbb{Z}$ consisting of N -vectors (n_1, \dots, n_N) with integer coefficients. This space is not a vector space (because \mathbb{Z} is not a field), in fact it is usually regarded as an Abelian group in which the "multiplication law" is just addition of integer vectors and the identity is the zero vector $(0, \dots, 0)$.

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[Note: Nakahara writes $\mathbb{Z} \oplus \mathbb{Z}$ instead of $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$. It is just the space of integer vectors (n_1, n_2) , a lattice in the plane.]

Don't confuse \mathbb{Z}^2 with \mathbb{Z}_2 ; the latter is the set $\{0, 1\}$ with addition modulo 2.
 \uparrow = set $\{(n_1, n_2) \mid n_1, n_2 \in \mathbb{Z}\}$.

Begin with another excursion into group theory. We'll only need Abelian groups for homology theory, but for now we'll consider some issues that apply to any group (Abelian or non-Abelian).

Recall that a group homomorphism is a map $f: G \rightarrow X$ between groups G and X such that $f(g_1)f(g_2) = f(g_1g_2)$, $\forall g_1, g_2 \in G$.

Def's: $\ker f = \{g \in G \mid f(g) = e_X\}$ $e_X =$ identity element in X
 $e_G =$ " " " " G .
 $\text{im } f = \{x \in X \mid x = f(g), \text{ some } g \in G\}$ (usual defn. of image).

Let $K = \ker f \subset G$
 $I = \text{im } f \subset X$. for brevity.

Thm: $\ker f = K$ is a normal subgroup of G .

Proof: ~~Let $k_1, k_2 \in K$.~~ First show K is a subgroup.

Let $k_1, k_2 \in K$. Then $f(k_1)f(k_2) = f(k_1k_2)$
 $= e_X e_X = e_X$,

so $k_1k_2 \in K$.

Similarly show K satisfies other axioms of a group.
 Next show that K is a normal subgroup. This means that

$gkg^{-1} \in K$ for all $g \in G$, $k \in K$. Easy:

$$\begin{aligned} f(gkg^{-1}) &= f(g)f(k)f(g^{-1}) = f(g)e_Xf(g^{-1}) \\ &= f(g)f(g^{-1}) = f(gg^{-1}) = f(e_G) = e_X. \end{aligned}$$

Hence $gkg^{-1} \in K$.

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Note: Should have pointed this out earlier, but if K is a normal subgroup of G , then the left cosets gK and right cosets Kg are identical (as subsets of G). This is what $gkg^{-1} \in K \forall g \in G, k \in K$ means. If a subgroup is not normal, then the left and right cosets are generally different.

Thm: $\text{im} f = I$ is isomorphic to the quotient group G/K ,

$$\frac{G}{\ker f} \cong \text{im} f. \quad (\text{The isomorphism is } [g] \mapsto f(g).)$$

Proof: First show that there is a 1-2-1 corresp. between cosets in G/K and elements of I . To do this we need to show that 2 elements g_1, g_2 of G map onto the same element of I iff they belong to the same coset of G/K . So suppose $f(g_1) = f(g_2)$. This means $f(g_1)f(g_2)^{-1} = e_X = f(g_1g_2^{-1}) \Rightarrow g_1g_2^{-1} \in K \Rightarrow g_1 = kg_2$ for some $k \Rightarrow g_1, g_2$ belong to same coset. Converse is easily proven. Thus $[g] \mapsto f(g)$ is a bijection.

Next we need to show that $[g] \mapsto f(g)$ is an isomorphism. This is easy. (That is, it's a homomorphism, ~~is~~ since we already know it's a bijection.)

Now specialize to Abelian groups. Note, for an Abelian group, every subgroup is normal, so the quotient group is always defined.

For Abelian groups, convenient to change notation, use "+" for "group multiplication" etc. Table:

general case		Abelian
xy	\longrightarrow	$x + y$
x^{-1}	\longrightarrow	$-x$
e	\longrightarrow	0
x^n	\longrightarrow	nx
x	\longrightarrow	\oplus (Cartesian product)