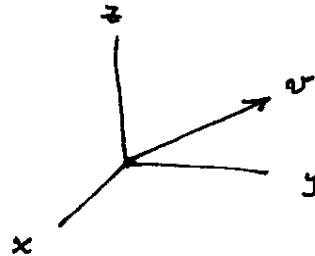


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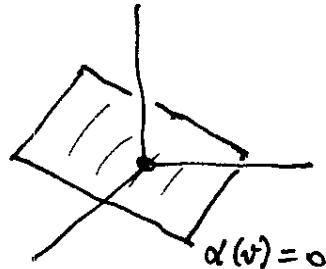
Remark: How to visualize a covector. To visualize a vector  $v \in V$  is easy, e.g. if  $V = \mathbb{R}^3$ ,



But what about  $\alpha \in V^*$ , i.e.  $\alpha: V \rightarrow \mathbb{R}$  (we'll take  $K = \mathbb{R}$  for this discussion).  $\alpha$  of course is a vector in  $V^*$ , but how can we visualize it in  $V$ ?

Well,  $\alpha$  is a real-valued function on  $V$ , so we can look at its contour surfaces (level sets),  $\alpha(v) = \text{const.}$ . The value 0 is particularly interesting, the level set  $\alpha(v) = 0$  is otherwise just the kernel of  $\alpha$ .

Can easily show that if  $\alpha \neq 0$ , then  $\ker \alpha$  is an  $(n-1)$ -dimensional vector subspace of  $V$  ( $\dim V = n$ ), i.e. a hyperplane passing thru the origin.



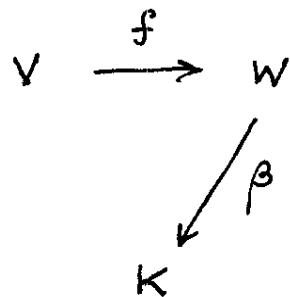
And the surfaces  $\alpha(v) = \text{const} \neq 0$  are other hyperplanes parallel to this one. So if you specify  $\ker \alpha$  (the hyperplane), and the value of  $\alpha$  on any parallel (but different) hyperplane, you have specified  $\alpha$ . These hyperplanes, especially  $\ker \alpha$ , are part of the geometrical interpretation of covectors.

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Begin with relation between dual spaces  $V^*$  and  $W^*$  when we have a map  $f: V \rightarrow W$ . (There is still no metric.) Does  $f$ , which takes a vector  $v \in V$  and produces a vector  $w = f(v) \in W$ , do something similar to  $V^*$  and  $W^*$ , that is, take a form (=dual vector) in  $V^*$  and produce another form in  $W^*$ ? The answer is no, in general. But it does allow one to take a form in  $W^*$  and produce another form in  $V^*$  (in the reverse direction from the action of  $f$  itself). This action on forms, going from  $W^* \rightarrow V^*$ , is called the pull-back.

Suppose we are given  $f: V \rightarrow W$  and some  $\beta \in W^*$ , that is  $\beta: W \rightarrow K$  (the scalars). Picture of the maps:



The picture makes it obvious that we can go directly from  $V$  to  $K$  by composing the maps, that is, let  $\alpha \in V^*$  be defined by

$$\alpha = \beta \circ f, \quad \alpha: V \rightarrow K.$$

This specifies a mapping between  $W^*$  and  $V^*$  which we denote by  $f^*: W^* \rightarrow V^*$ , called the pull-back of  $f$ . That is,

$$f^*: W^* \rightarrow V^*: \beta \mapsto \beta \circ f.$$

An equivalent definition of the pull-back is to specify  $f^* \beta$  by its action on vectors in  $V$ :

$$(f^* \beta)(v) = \beta(f(v)), \quad \forall v \in V \quad \text{defines } f^* \beta.$$

There is no natural way to define a map:  $V^* \rightarrow W^*$  (a "push-forward") unless  $f$  is invertible, whereupon you could use  $f^{-1*}$ .

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We've done almost everything that can be done without a metric, so now let's introduce one. Nakahara's discussion of this is backwards, confused, and wrong in part, so ignore what he says and use the following.

Begin with the case  $K = \mathbb{R}$  (real vector spaces), since things are somewhat more complicated when  $K = \mathbb{C}$ . Idea of metric is measure of distance.

Given real vector space  $V$ . A metric or (metric tensor) is a map

$$g: V \times V \rightarrow \mathbb{R},$$

such that:

- 1)  $g$  is linear in both operands,

$$\left. \begin{aligned} g(c_1 u_1 + c_2 u_2, v) &= c_1 g(u_1, v) + c_2 g(u_2, v) \\ g(u, c_1 v_1 + c_2 v_2) &= c_1 g(u, v_1) + c_2 g(u, v_2) \end{aligned} \right\} \begin{array}{l} \forall u_1, u_2, u \in V \\ \forall v_1, v_2, v \in V \\ \forall c_1, c_2 \in \mathbb{R}. \end{array}$$

- 2)  $g$  is positive definite,

$$g(v, v) \geq 0, \quad \forall v \in V$$

$$g(v, v) = 0 \text{ iff } v = 0$$

- 3)  $g$  is symmetric,

$$g(u, v) = g(v, u), \quad \forall u, v \in V.$$

Then the quantity  $g(u, v)$  is the inner product of 2 vectors, which we may denote by  $\langle u, v \rangle$  (another notation for it).

Let  $\{e_i\}$  be a basis in  $V$ . Then we define

$g_{ij} = g(e_i, e_j) = \text{component matrix of } g \text{ in the given basis.}$

Condition 2 implies that  $g_{ij}$  is a positive definite matrix, hence that  $\det g_{ij} \neq 0$  ( $g_{ij}$  is nonsingular, since all of its eigenvalues are positive.)

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Note that in relativity theory, we deal with metrics that are not positive definite. In this case, we replace requirement 2) in the definition with the nonsingularity requirement, that  $\det g_{ij} \neq 0$ . (This condition makes reference to a basis, but is independent of the basis chosen.) A way of writing the nonsingularity condition without reference to a basis is to say, ~~if~~ if  $g(u, v) = 0$  for all  $u \in V$ , then  $v = 0$ .

A metric, regarded as a distance function on  $V$ , induces an association between  $V$  and  $V^*$ . The latter is an alternative way of looking at a metric. In the expression  $g(u, v)$ , regard  $u$  as fixed and  $v$  as variable. To emphasize this, write

$$g_u(v) = g(u, v),$$

thereby defining a function  $g_u : V \rightarrow \mathbb{R}$ . Such a function is a form, i.e.,  $g_u \in V^*$ . Thus we have a ~~phi~~ mapping, \*

$$g : V \rightarrow V^* : u \mapsto g_u. \quad (\text{linear})$$

It's abuse of notation to use the same symbol  $g$  for this map as for the distance function, but they're so closely related that everyone does so anyway. Now put this into component language.

~~Let  $v \in V$ , let  $\{e_i\}$  be a basis in  $V$  so that  $v = \sum_{i=1}^n v^i e_i$ , and let  $\alpha = g_v$ .~~

Let  $u \in V$ , let  $\{e_i\}$  be a basis in  $V$  and write  $u = \sum_{i=1}^m u^i e_i$ , and let  $\alpha = g_u \in V^*$ . Find components  $\alpha_i$  of  $\alpha$  w.r.t. the dual basis.

$$\alpha(v) = \sum_j \alpha_j v^j = g_u(v) = \sum_{i,j} u^i g_{ij} v^j, \quad \forall v \in V$$

$$\text{so } \alpha_i = \sum_j u^i g_{ij}.$$

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The form  $\alpha = g_{ij} u^i$  is so closely associated with  $u^i$  that in applications it is often identified with it, and we just write  $u_i$  (with a lower index) instead of  $\alpha_i$  or  $(g_{ij})_i$ . This is called lowering an index.

$$u_i = \sum_j g_{ij} u^j.$$

Now because  $g_{ij}$  is invertible, we ~~can~~ have the inverse map:  $V^* \rightarrow V$ .

Let  $g^{ij}$  be the inverse matrix of  $g_{ij}$  (std notation), so that

$$\sum_k g_{ik} g^{kj} = \delta_i^j.$$

Then the inverse map  $g^{-1}: V^* \rightarrow V: \alpha \mapsto u$  is specified in components by

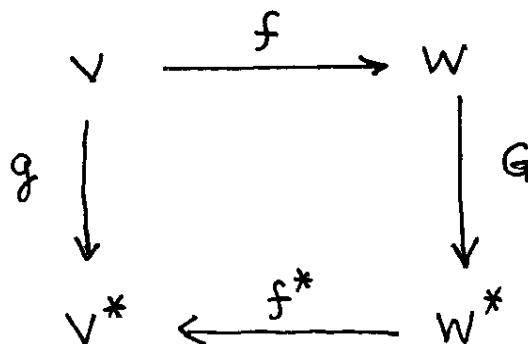
$$u^i = \sum_j g^{ij} \alpha_j.$$

One often writes  $\alpha^i$  instead of  $u^i$  (same symbol, but with an upper index) and speaks of raising an index.

Summary: A metric  $g$  on  $V$  induces an isomorphism between  $V$  and  $V^*$ .

$\swarrow$  real vector space       $\square$  linear

Now examine how metrics interact with maps. Suppose we have a linear map between spaces, each of which possesses a metric. Let  $g$  be the metric on  $V$  and  $G$  the metric on  $W$ , and suppose  $f: V \rightarrow W$  is linear. Then we have the following picture of maps and spaces,



Notice that there is a route to get from  $W$  to  $V$ , since  $g$  and  $G$  are invertible.

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Leads to definition,  $\tilde{f}: W \rightarrow V$ , the adjoint of  $f$ , by

$$\tilde{f} = g^{-1} f^* G. (= g^{-1} \circ f^* \circ G).$$

This is equivalent to

$$g(\tilde{f}w, v) = G(w, fv), \quad \forall v \in V, w \in W,$$

or

$$\langle \tilde{f}w, v \rangle_g = \langle w, fv \rangle_G, \quad \text{which should look familiar.}$$

(exercise to show this).

Now what changes when you go to metrics on complex vector spaces.

Now  $g: V \times V \rightarrow \mathbb{C}$ , such that:

1)  $g$  is linear in 2nd operand, and anti-linear in the first,

$$g(c_1 u_1 + c_2 u_2, v) = \bar{c}_1 g(u_1, v) + \bar{c}_2 g(u_2, v)$$

$$g(u, c_1 v_1 + c_2 v_2) = c_1 g(u, v_1) + c_2 g(u, v_2)$$

2)  $g(v, v) = \text{real}, > 0, \quad \forall v \in V$

$$g(v, v) = 0 \quad \text{iff } v = 0$$

$$3) \quad g(u, v) = \overline{g(v, u)},$$

(overbar = complex conjugate).

The associated mapping  $g: V \rightarrow V^*$  is defined as before,

$$u \mapsto g_u, \quad g_u(v) = g(u, v) = \langle u, v \rangle,$$

but it is now an antilinear map (point missed by Nakahara).

The adjoint is defined as above,  $\tilde{f} = g^{-1} f^* G$ . Note that it is linear, since  $g^{-1}$  and  $G$  are antilinear. (Usual notation in QM,  $\tilde{f} = f^+$ ).

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Note: usually in QM when we talk about the adjoint of a linear operator, we are thinking of the case  $W=V$ , so  $G=g$ , and so  $\tilde{f} = g^{-1}f^*g$ , and

$$\langle \tilde{f}u, v \rangle = \langle u, fv \rangle.$$

Next, tensors. ( $K=\mathbb{R}$  here).

A tensor  $T$  of type  $(p,q)$  is a multilinear map,

$$T: \underbrace{V^* \times \dots \times V^*}_{p \text{ times}} \times \underbrace{V \times \dots \times V}_{q \text{ times}} \rightarrow \mathbb{R}$$

Examples:

A covector  $\alpha \in V^*$  is a tensor of type  $(0,1)$ , since  $\alpha: V \rightarrow \mathbb{R}$ .

A vector  $v \in V$  is considered a tensor of type  $(1,0)$ , since it can be considered a map  $v: V^* \rightarrow \mathbb{R}: \alpha \mapsto \alpha(v) \equiv v(\alpha)$ .

$g: V \times V \rightarrow \mathbb{R}$  is a tensor of type  $(0,2)$

$g^{-1}: V^* \times V^* \rightarrow \mathbb{R}$  " " " " "(2,0)

etc.

There are 2 operations on tensors, the tensor product and the contraction.

Let  $\mu = \text{tensor of type } (p,q)$   
 $\nu = \text{ " " " } (r,s)$

then  $\mu \otimes \nu = \text{ " " " } (p+r, q+s)$ .

$$\begin{aligned} \underline{\text{Def: }} \quad & \mu \otimes \nu (\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_r; v_1, \dots, v_q, u_1, \dots, u_s) \\ &= \mu(\alpha_1, \dots, \alpha_p; v_1, \dots, v_q) \nu(\beta_1, \dots, \beta_r; u_1, \dots, u_s). \end{aligned}$$

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The contraction takes a tensor of type  $(p, q)$  and produces one of type  $(p-1, q-1)$ . It does not require a metric for its definition. We illustrate in case of contraction on first slots. Let  $\mu$  = a tensor of type  $(p, q)$ .

$$(\text{contracted } \mu)(\alpha_2, \dots, \alpha_p; v_2, \dots, v_q) = \sum_i \mu(e^{i*}, \alpha_2, \dots, \alpha_p; e_i, v_2, \dots, v_p)$$

where  $\{e_i\}$  is a basis in  $V$  and  $\{e^{i*}\}$  is the dual basis in  $V^*$ .

The contraction is independent of the basis (but it does depend in general on which slots are contracted).

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Now some background on spaces, manifolds, topology, etc.  
First some overview of the hierarchy of spaces you get as you add more and more structure.

At the most primitive level, a "space" is just a set of objects we call "points" without any additional structure.

To this we may add the structure required to give the space a topology. Intuitively, topology tells you something about which points are "close" to one another, so that you can think about "neighborhoods" of points and how different neighborhoods fit together to make the whole space.

Naively, it would seem that the concept of "closeness" requires a definition of "distance" between points, but this is not so. Instead, all one needs is a definition of the open subsets of the space (if it is to be a topological space). Here is the official definition:

A topological space is a set  $X$  plus a set  $\{U_i\}$  of subsets, the "open subsets", such that

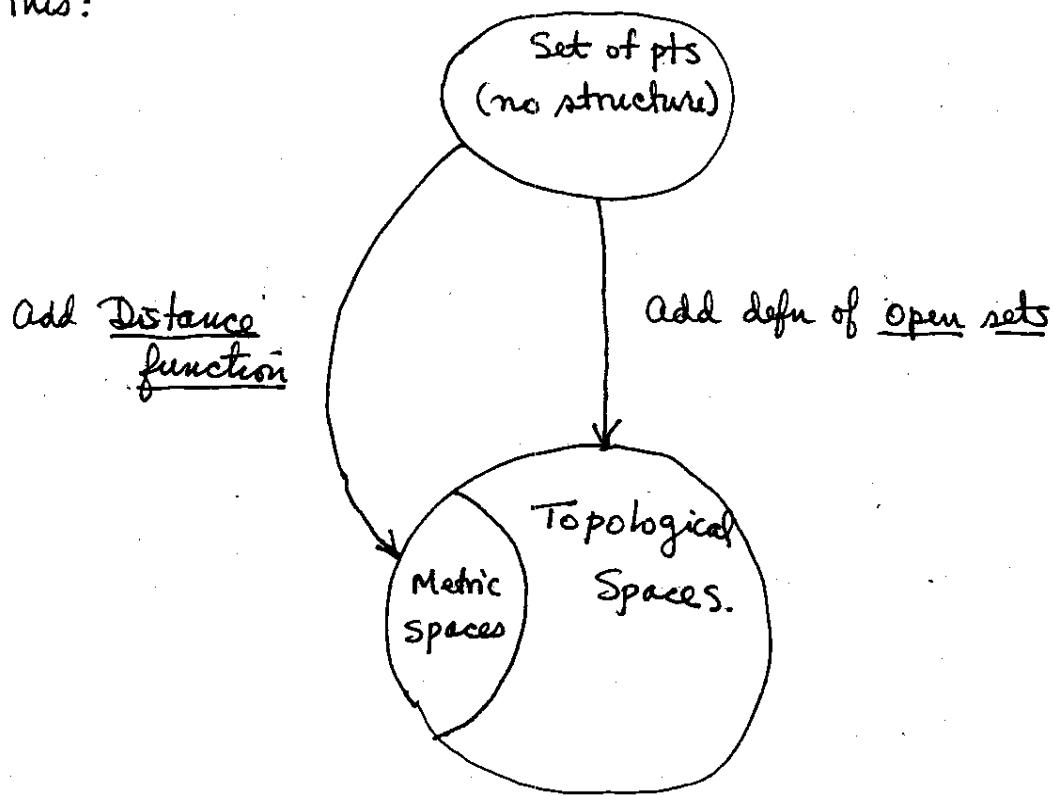
- 1) Both  $X$  and  $\emptyset$  (the empty set) are in  $\{U_i\}$
- 2) The union of any number (finite or infinite) of open subsets is open (i.e., it is in  $\{U_i\}$ )
- 3) The intersection of any finite number of open sets is in  $\{U_i\}$ .

We won't work at the level of this definition very much, but

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it is worthwhile to have it.

On the other hand, if we take our space  $X$  and introduce a distance function, then we obtain a metric space, and open sets can be defined in the usual way using the concept of distance. For example, an "open ball" centered at  $x_0$  is the set of all points  $x$  such that  $d(x_0, x) < r$  for some radius  $r > 0$ , where  $d(\cdot, \cdot)$  is the distance function. Thus, all metric spaces automatically become topological spaces (they form a subset of all topological spaces). (The distance function is not to be confused with a metric tensor; the latter can be used to construct a distance function, but that is just a special case.) Thus we have a diagram like this:



The distance function has to satisfy certain requirements.

But since topological matters can be expressed purely in terms of

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the open sets (this is the more primitive concept), we use that in developing topology, not distances.

For example, let  $f: X \rightarrow Y$  be a map between two topological spaces. Then we say that  $f$  is continuous if the inverse images of open sets (in  $Y$ ) are open sets (in  $X$ ). This is the topological definition of continuity. It coincides with usual  $\epsilon$  and  $\delta$  definition in the case of a metric space, but is more general.

The definition of topology leaves open the possibility that a given set  $X$  can be endowed with a topology in more than one way, ~~and~~ (that is, you define open sets in more than one way), and this is true.

In the case of  $X = \mathbb{R}$ , the usual topology is given by defining open sets to be open intervals and their unions and finite intersections. This is the same as the topology given by the distance function  $d(x, y) = |x - y|$ ,  $x, y \in \mathbb{R}$ . The usual topology on  $\mathbb{R}^n$  is defined similarly. In this course we will only use the usual topology. We can also define the usual topology on subsets of  $\mathbb{R}^n$ , which covers all the spaces we will use in this course.

The usual topology has the Hausdorff property. See book for definition. We will use the usual topology, so all our spaces will have the Hausdorff property (usual in physical applications).