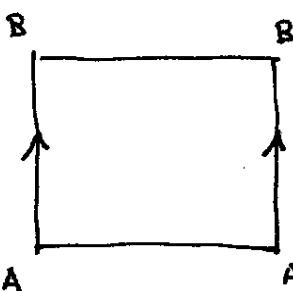


(1)
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Most of the examples so far involve equivalence relations that are produced by group actions. (More on group actions later.) Here are some equivalence relations that are not produced by group actions.

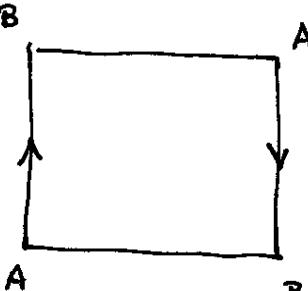
(7) $X = \text{square with edges identified.}$ (Equivalence classes imply glueing rules.)

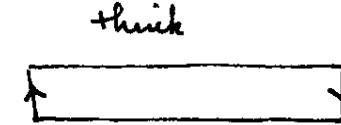


$$= \frac{X}{\sim} =$$

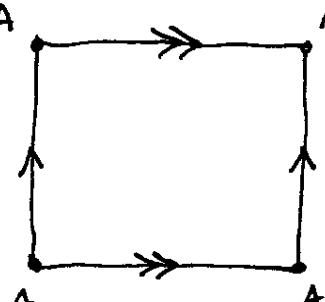

segment of cylinder.

(Each point $x \in X$ is in an equivalence class by itself, except for points on the right and left sides, which are identified in pairs.) Variations on this:



$$= \text{Möbius strip}$$


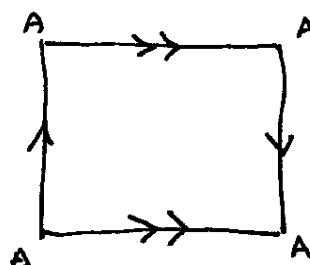
think



$$=$$

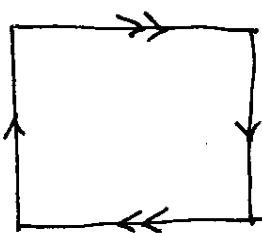

$$= 2\text{-torus}, T^2$$

\Leftarrow points on corners are in a 4-point equiv. class. Other points on sides are in 2-point equiv. classes. Other (interior) points are in 1-point equiv. classes.

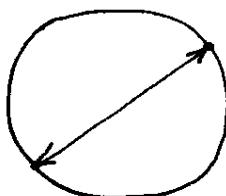
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= Klein bottle.

a 2-dim. surface that
cannot be imbedded in \mathbb{R}^3
without self-intersections.



=

= \mathbb{RP}^2

disk with opposite
points on boundary
identified

Note: the 2-disk D^2 is the set $x^2+y^2 \leq 1$ in the plane, i.e. it is the interior of a circle plus the boundary.

x-y

(B) A different identification for the disk D^2 . Let interior points be in equivalence classes, let all boundary points be in a single equiv. class.

$$X = D^2 = \text{circle with diagonal lines} . \quad \text{Let } x \sim y \text{ if } |x| = |y| = 1.$$

Then

$$\frac{X}{\sim} = \frac{D^2}{\sim}, \quad \text{circle with diagonal lines} \rightarrow \text{half-disk} \rightarrow \text{circle} = S^2.$$

$$\frac{D^2}{\sim} = S^2.$$

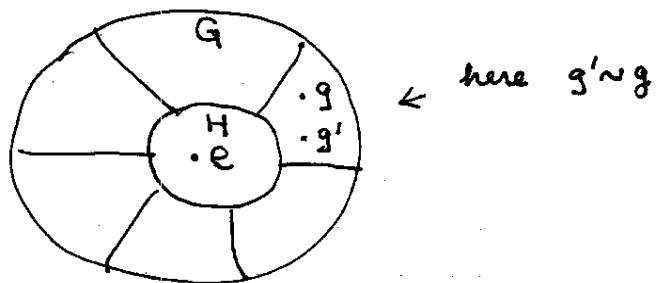
(3)

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Another example of equiv. classes, quotient spaces, this time from abstract group theory. Let G = a group, H a subgroup; $H \subset G$. Let $g, g' \in G$, and define $g' \sim g$ if $\exists h \in H$ such that $g' = gh$. Easy to show from defn that this is an equivalence relation; can also show this by using group actions, as in HW problem. Now identify equivalence classes by representative elements. Start with $[e]$, which is

$$[e] = \{eh \mid h \in H\} = H.$$

The equiv. class of the identity is the subgp. H itself. But since equiv. classes are disjoint, G gets divided up like this:



* Consider $[g]$ for any element $g \in G$. This is the set

$$[g] = \{gh \mid h \in H\} = gH$$

where the notation gH means, multiply each element of H by g to get a new list of elements of G , which constitute the equiv. class $[g]$. This equiv. class is called a left coset (of H).

One can also define right cosets. These are the sets

$$\{hg \mid h \in H\} = Hg,$$

and they correspond to a different equivalence relation, $g' \sim g$

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when $g' = hg$ for some $h \in H$. In general, the right cosets are different from the left cosets (the two equivalence relations divide G into equiv. classes in two different ways).

Return to left cosets. We can define the quotient space

$$\frac{G}{\sim} = \frac{G}{H} = \text{space of left cosets} = \underline{\text{coset space}}$$

↑ this is one notation for it.

In using notation like this, the equiv. relation intended must be supplied by context (left cosets, right etc.). Here we are thinking of left cosets. The quotient space is also called a "homogeneous space." It is a space in which each point represents a coset in G .

Does G/H inherit a group structure from G ? That is, is it meaningful to multiply cosets? If $a, b \in G$ and $[a], [b]$ are cosets, then the logical definition of the product of two cosets is

$$[a][b] = [ab].$$

But this is only meaningful if the equivalence class on the RHS is independent of the representative elements a, b on the LHS. That is, let $a' = ah_1, b' = bh_2$ where $h_1, h_2 \in H$, so that a', b' are new representative elements on the LHS. Is it true that $a'b' \sim ab$, i.e. that $a'b' = abh_3$, some $h_3 \in H$, i.e. that $(ab)^{-1}a'b' \in H$? But

$$(ab)^{-1}a'b' = b^{-1}a^{-1}ah_1bh_2 = b^{-1}h_1bh_2,$$

and this $\in H$ iff $b^{-1}h_1b \in H$. Thus we can say, the proposed multiplication law of cosets is meaningful if for all

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$b \in G, h \in H, b^{-1}hb \in H$. But this implies 9/108
 $b^{-1}Hb = H$ (with the set theoretic interpretation of multiplying
a group element times H). This leads to a definition:

Def. A subgroup $H \subset G$ is said to be normal if $b^{-1}Hb = H$,
 $\forall b \in G$.

This does not mean that each individual element of H is invariant under conjugation by $b \in G$, just that if you conjugate all the elements of H by b , you get the same list all over again, perhaps in a different order. As we say, H is an invariant subgroup under conjugation by any element of G .

So the logic above says that multiplication of cosets is defined if H is normal. The converse is also true.

The quotient (or coset) space $\frac{G}{H}$ is defined as a set regardless of the properties of H , but it is a group (the quotient group) iff H is normal.

If H is normal, then the left cosets and right cosets are identical, so it doesn't matter which equiv. relation is used to construct the quotient group. If H is not normal, then the 2 spaces of cosets (left and right) are not the same.

If G is Abelian, then $b^{-1}Hb = Hb^{-1} = H$, and all quotient spaces G/H are groups (all subgroups are normal).

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Now for some selected topics in linear algebra, with a slight geometrical flavor.

Let V be a vector space over a field K . In practice, K usually = \mathbb{R} or \mathbb{C} (real or complex vector spaces); this means that the coefficients used in forming linear combinations of vectors (scalars) are either real or complex numbers. (In general it is not meaningful to say whether the vectors themselves are "real" or "complex".)

Assume you know definitions of vector space, basis, linear (in)dependence, span, dimension, etc. We will (for now) deal only with finite dimensional vector spaces.

Psychological problem: Physicists have a tendency to assume that all vector spaces possess a metric, i.e., a definition of a scalar product, but for many problems the vector spaces you encounter do not possess any metric that is ~~a~~ natural to the problem at hand. (Example: the (x, p) phase space of a mechanical problem in 1D.) Of course you can always introduce an arbitrary metric, but this is almost always a bad idea unless the metric is a natural outcome of the structure presented by your problem, or ^(occasionally) unless you can show that results don't depend on the choice of the metric.

Instead it is better to develop those structures of linear algebra that can be developed without reference to any metric, and to understand those. Then we introduce a metric and see what new structures become available. This is what we shall do.

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First, let V be a vector space and $\{e_1, e_2, \dots, e_n\}$ be a basis in V (thus, $\dim V = n$). If $v \in V$, then we can write in a unique way,

$$v = \sum_{i=1}^n v^i e_i,$$

where $v^i \in K$ are the components of v w.r.t. the basis $\{e_i\}$. The upper (contravariant) index on v is deliberate. Note, $v^i \in K$ but $e_i \in V$.

Now consider linear maps, that is, linear homomorphisms between vector spaces, $f: V \rightarrow W$. These satisfy

$$\begin{aligned} f(v_1 + v_2) &= f(v_1) + f(v_2), & \forall v_1, v_2 \in V. \\ f(kv) &= kf(v) & \forall v \in V, k \in K \end{aligned}$$

↑ the field

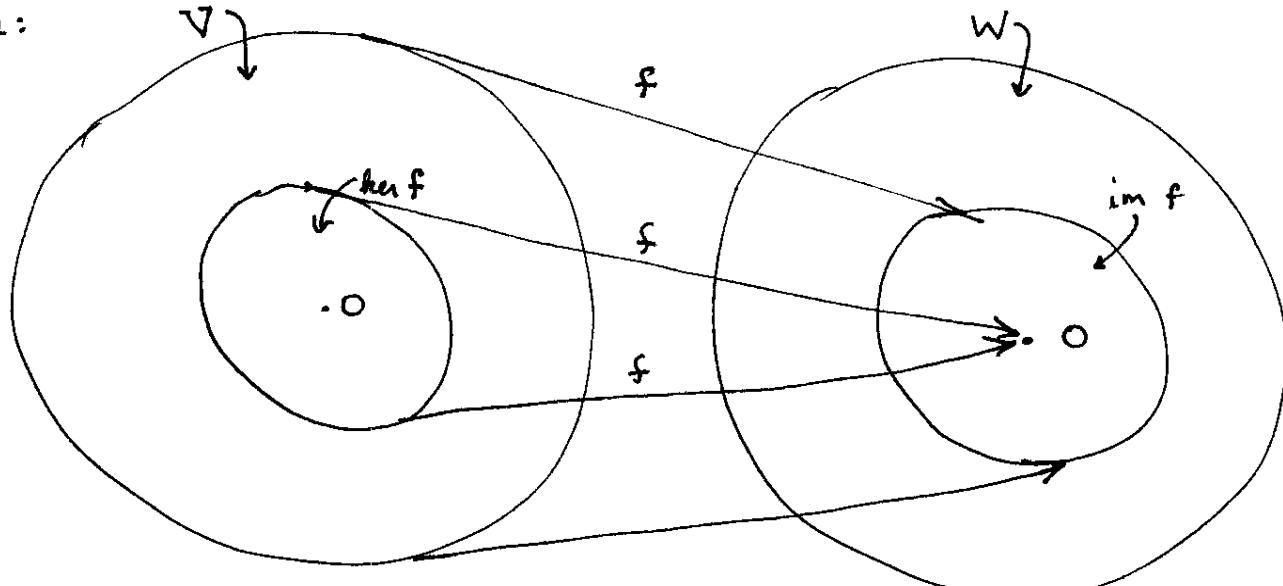
There are two spaces that can be defined using only the linear structure:

$\ker f = \{v \in V \mid f(v) = 0\}$ = set of all vectors in V annihilated by f

$\text{im } f = \{w \in W \mid w = f(v) \text{ for some } v \in V\}$ = usual defi. of image.

Note, $\begin{cases} \ker f \subset V \\ \text{im } f \subset W \end{cases}$, note also, $\ker f$ is a vector subspace of V
 $\text{im } f$ is a vector subspace of W

Picture:



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Basic theorem (important):

$$\dim \ker f + \dim \text{im } f = \dim V$$

Context: $f: V \rightarrow W$, linear map.
field K ($= \mathbb{R}$ or \mathbb{C})

Proof: Let $\{g_1, \dots, g_r\}$ be a basis for $\ker f$ ($g_i \in V$)

Let $\{h'_1, \dots, h'_s\}$ be a basis for $\text{im } f$ ($h'_i \in W$).

Let $v = \text{any vector in } V$. Note $f(v) \in \text{im } f$, so it can be expanded in the basis $\{h'_i\}$:

$$f(v) = \sum_{i=1}^s c^i h'_i, \quad c^i = \text{coefficients}, \quad c^i \in K.$$

But for each i , \exists some vector $h_i \in V$ such that $h'_i = f(h_i)$ ($f: h_i \mapsto h'_i$)
(h_i is not unique, in general). So

$$f(v) = \sum_{i=1}^s c^i f(h_i) = f\left(\sum_{i=1}^s c^i h_i\right),$$

$$\text{or } f\left(v - \sum_{i=1}^s c^i h_i\right) = 0. \quad \text{Thus } v - \sum_{i=1}^s c^i h_i \in \ker f,$$

and this vector can be expanded in the basis $\{g_i\}$, say $\sum_{i=1}^r d^i g_i$, $d^i \in K$

$$\text{thus, } v = \sum_{i=1}^r d^i g_i + \sum_{i=1}^s c^i h_i,$$

and we see that any $v \in V$ can be written as a lin. comb. of $\{g_i\}, \{h_i\}$.

Now, are these vectors lin. indep? To show that they are, let

$$\sum_{i=1}^r a^i g_i + \sum_{i=1}^s b^i h_i = 0.$$

Apply f to both sides, use fact that $f(g_i) = 0$, $f(h_i) = h'_i$. Gives

$$\sum_{i=1}^s b^i h'_i = 0 \Rightarrow b^i = 0 \text{ since } \{h'_i\} \text{ are lin. indep.}$$

But this $\Rightarrow \sum_{i=1}^r a^i g_i = 0 \Rightarrow a^i = 0$ since the $\{g_i\}$ are lin. indep. So all

coeffs a^i, b^i vanish, and the set $\{g_i, h_i\}$ are lin. indep. and span V
(they form a basis for V).

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Thus, $\dim V = r+s = \dim \ker f + \dim \text{im } f$. QED

Some intuition about this theorem: f acts on V and annihilates some vectors (those in $\ker f$). The ones it doesn't annihilate go into $\text{im } f$. Therefore $\dim V = \dim \ker f + \dim \text{im } f$. (At least this is a way of remembering the theorem.)

Remark: Nakahara calls the subspace of V spanned by the $\{h_i\}$ the "orthogonal complement" to $\ker f$, and he writes $(\ker f)^\perp$ for it. Please ignore this. We don't have a metric on our vector spaces yet, so "orthogonal" is undefined. In any case, the vectors $\{h_i\}$ are not unique, because you can add any element of the kernel to them without changing their definition. That is, if $k \in \ker f$, then $f(h_i+k) = f(h_i) = h_i$, so h_i+k works just as well as h_i .

So the "space spanned by the $\{h_i\}$ " has no invariant meaning.

Another remark: It turns out, however, that there is a way of defining a space that is "complementary" to $\ker f$, in a certain sense, but it is not a subspace of V , it is a quotient space. See the HW problems.

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Now we move to the concept of the dual space to a vector space V . Setup:

Let K be the field over which V is defined ($K = \mathbb{R}$ or \mathbb{C} in practice). Elements of K are called "scalars". We assume V is finite-dimensional.

Let α be a linear map, $\alpha: V \rightarrow K$. This is a special case of a

Such maps are called dual vectors or co-vectors.

linear map, where $W = K$
(a one-dimensional space).

Def. The dual space to V , denoted V^* , is the set of all such linear maps:
(dual vectors)

$$V^* = \{\alpha \mid \alpha: V \rightarrow K, \text{ linear}\}.$$

First note that V^* (like V) is a vector space, under the obvious definition of multiplication of maps by scalars and the addition of maps:

For $\alpha_1, \alpha_2 \in V^*$ and $c_1, c_2 \in K$, define $c_1\alpha_1 + c_2\alpha_2$ by

$$(c_1\alpha_1 + c_2\alpha_2)(v) = c_1\alpha_1(v) + c_2\alpha_2(v), \quad v \in V.$$

Now it turns out that $\dim V = \dim V^*$ (important fact). Easy way to see this:

Let $\dim V = n$, let $\{e_i, i=1, \dots, n\}$ be a basis in V .

Define $\alpha_i = \alpha(e_i)$, call $\{\alpha_i\}$ the components of $\alpha: V \rightarrow K$.

For given α , the components α_i uniquely specify α . That is, if the n scalars $\alpha_i \in K$ are given, then the action of α on any vector $v \in V$ is determined:

$$\alpha(v) = \alpha\left(\sum_{i=1}^n v^i e_i\right) = \sum_{i=1}^n v^i \alpha(e_i) = \sum_{i=1}^n v^i \alpha_i$$

Conversely, if α is given, the α_i are determined by $\alpha_i = \alpha(e_i)$.

So this provides a 1-to-1 map between V^* and K^n ($= \mathbb{R}^n$ or \mathbb{C}^n).

Moreover, this map is linear (hence a linear isomorphism). Hence $\dim V^* = n = \dim V$.

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Remark 1: The lower index on α_i is deliberate, just as is the upper index on v^i : the lower index is a "covariant" index and the upper is "contravariant". (Standard terminology in tensor analysis.)

Remark 2: The concept of the dual space is important. Often the analysis of a problem becomes clarified when we switch attention from some space to its dual. This will be an important theme in this course.

Return to dual space. We have chosen a basis $\{e_i\}$ in V . What about a basis in V^* ? This would be a set of n linearly independent covectors, call them $\{e^{*i}, i=1, \dots, n\}$, which can be defined by specifying their action on the basis $\{e_i, i=1, \dots, n\}$ in V . The definition

$$e^{*i}(e_j) = \delta_j^i$$

is convenient. It means

$$\alpha = \sum_i \alpha_i e^{*i},$$

where $\alpha_i = \alpha(e_i)$ are the components as defined above. Thus those components are the components in the usual sense of α w.r.t. the basis $\{e^{*i}\}$.

The basis $\{e^{*i}\}$ in V^* is said to be dual to the basis $\{e_i\}$ in V .

"Inner product" notation mentioned by Nakano. N. wants to use the notation,

$$\langle \alpha, v \rangle = \alpha(v)$$

and call it an "inner product." This is ok as long as you don't confuse this with the inner product associated with a metric (which we haven't introduced yet). This \langle , \rangle operation is a map,

$$\xrightarrow{\quad} \langle , \rangle : V^* \times V \rightarrow K : (\alpha, v) \mapsto \alpha(v).$$

But to be safe I'd prefer not to call this an "inner product" because we will define an inner product later that does involve the metric.

(different)