Physics 250
Fall 2008
Homework 4
Due Friday, September 26, 2008

Reading Assignment: Nakahara, pp. 121–131, 153–167. I also recommend Frankel, pp. 567–575.

Notes. Here are some comments on the text, which is confusing in several places.

On p. 130, in the paragraph that begins with "A retract is not necessarily ...," insert the following punctuation, and the text will be more clear. Put [...] around the first two sentences of this paragraph, then start a new paragraph with "Since X and R ...." The bracketed text is a digression, and with the sentence "Since X ..." he is returning to the topic of a deformation retract. Thus, the X and R referred to in this sentence are not the X and R in Fig. 4.8, but rather any X and R which are related by the deformation retract (for example, the sets in Fig. 4.7). In fact, for the X and R of Fig. 4.8, Eq. (4.7) is not true. (This part of the text was confusing in the first edition, but Nakahara made it worse in the second edition by combining two paragraphs.)

Nakahara's proof that $\pi_1(S^1) = \mathbb{Z}$ on pp. 131–133 is pretty hard to follow. It will be done better in class, in which the concepts of *covering space* and *lift of a curve* will be explained.

In Theorem 4.6 on p. 133 (Theorem 4.24 on p. 101 of the first edition), he should use the symbol $\times$ instead of $\oplus$. He is talking about the Cartesian product of groups, explained in class.

Note: This homework has 5 problems, but you only need to do 4 to get 100% credit (or 2 for 50%, etc). You choose which ones you want to do.

1. Nakahara Exercise 4.9, p. 160 (Exercise 4.53, p. 124 of first edition).

2. Nakahara, problem 4.1, p. 167 (problem 4.1, p. 128 of first edition).

3. (DTB) Nakahara, problem 4.2, p. 168 (problem 4.2, p. 128 of first edition).

4. (DTB) This problem concerns the relationship between the "classical" rotation group $SO(3)$ and the spin rotation group $SU(2)$. It is a special case of a space M and its *covering space* $\bar{M}$ (here $M = SO(3)$ and $\bar{M} = SU(2)$).

(a) Study the geometrical and intuitive argument that shows that $SO(3) = \mathbb{R}P^3$ (here = means, "is homeomorphic to"). Make sure you understand the interpretation of $\mathbb{R}P^3$ as the "northern hemisphere" of S^3 with antipodial points on the "equator" identified (and note that the "equator" is S^2).

Now show that $SU(2) = S^3$. Do it this way. Write an arbitrary 2×2 complex matrix as

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{1}$$

where $a, b, c, d \in \mathbb{C}$. Then show that if $UU^\dagger = U^\dagger U = 1$ and $\det U = 1$, then $c = -b^*$ and $d = a^*$, and $|a|^2 + |b|^2 = 1$. (Here we write simply 1 for the 2×2 identity matrix.) Now break a and b into real and imaginary parts by writing $a = x_0 - ix_3$, $b = -x_2 - ix_1$, so that

$$U = x_0 - i\mathbf{x} \cdot \boldsymbol{\sigma}, \tag{2}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1. \tag{3}$$

This shows that every element $U \in SU(2)$ can be associated with a unique point on S^3, whose coordinates in the $\mathbb{R}^4$ space in which the unit S^3 sphere is imbedded are the *Cayley-Klein* parameters (x_0, x_1, x_2, x_3). Conversely, show that every point on S^3 is associated with a unique element $U \in SU(2)$. The mapping between $SU(2)$ and S^3 is thus one-to-one. Part of this was done in lecture.

(b) In class we discussed the projection map, $p : SU(2) \to SO(3)$, given explicitly by

$$R_{ij} = \tfrac{1}{2} \operatorname{tr}(U^\dagger \sigma_i U \sigma_j), \tag{4}$$

where $R = p(U)$. Show that this is a group homomorphism. To do this, note that for any 2×2 matrix M we have

$$M = \tfrac{1}{2} \operatorname{tr}(M) + \tfrac{1}{2} \sum_{i=1}^{3} \sigma_i \operatorname{tr}(\sigma_i M). \tag{5}$$

Note that $p(U) = p(-U)$. It can be shown that $\operatorname{img} p = SO(3)$ (p is onto). What is $\ker p$?

(c) Let an angular velocity vector be defined as a function of time by

$$\boldsymbol{\omega}(t) = -g \frac{e}{2mc} \mathbf{B}(t), \tag{6}$$

where $\mathbf{B}$ is a magnetic field, a given function of time. Then the Schrödinger equation for the evolution of a spin-$\frac{1}{2}$ particle in the given magnetic field is

$$i \frac{d\psi}{dt} = \boldsymbol{\omega}(t) \cdot \frac{\boldsymbol{\sigma}}{2} \psi, \tag{7}$$

where $\psi \in \mathbb{C}^2$. The expectation value of the spin,

$$\mathbf{S}(t) = \langle \psi | \frac{\hbar}{2} \boldsymbol{\sigma} | \psi \rangle \tag{8}$$

obeys the equation

$$\frac{d\mathbf{S}}{dt} = \boldsymbol{\omega}(t) \times \mathbf{S}. \tag{9}$$

Since (7) is a linear equation, its solution can be written

$$\psi(t) = U(t)\psi(0), \tag{10}$$

for some 2×2 complex matrix U, satisfying $U(0) = 1$. Show that $U(t) \in SU(2)$.

Since (9) is a linear equation, its solution can be written

$$\mathbf{S}(t) = R(t)\mathbf{S}(0), \tag{11}$$

for some 3×3 real matrix R, satisfying $R(0) = I$. Show that $R(t) \in SO(3)$.

Now show that $R(t) = pU(t)$, where p is the projection : $SU(2) \to SO(3)$ defined by (4).

5. Consider the following theorem: If X and Y are topological spaces of the same homotopy type, and if $f : X \to Y$ is the map entering into the definition of "same homotopy type", then

$$\pi_1(X, x_0) \simeq \pi_1(Y, y_0), \tag{12}$$

where $y_0 = f(x_0$ and where "$\simeq$" means, "is isomorphic to."

A partial proof of this theorem was given in class (and in the notes), where a map

$$K : \pi_1(X, x_0) \to \pi_1(Y, y_0) \tag{13}$$

was constructed. Complete the proof.