Physics 250
Fall 2008
Homework 15
Due Friday, December 12, 2008

Reading Assignment: Nakahara, pp. 374–385.

1. (DTB) Nakahara problem 9.2, p. 372 (problem 2, p. 328, of the first edition).

2. (DTB) A problem on the Frobenius theorem. As discussed in class, a k-*distribution* over a manifold M ($\dim M = m$) is an assignment of k-dimensional subspaces ($k \le m$) in the tangent spaces T_xM for all $x \in M$ (or perhaps only over some region $U \in M$). The assignment is assumed to be smooth. We say that a k-distribution (in some region) is *integrable* if there exists an $(m-k)$-parameter family of k-dimensional surfaces in the region which are everywhere tangent to the distribution. The surfaces are only required to exist locally.

Let X, Y be vector fields on M which lie in a k-distribution Δ. If Δ is integrable, then the integral curves of both X and Y must lie in the surfaces tangent to Δ. Therefore if we follow the integral curves of X or Y in any order for any elapsed parameters, we must always remain on a given surface. Considering infinitesimal elapsed parameters, we see that the Lie bracket $[X, Y]$ must be tangent to the surface, that is, it must lie in the distribution. Thus, we have a theorem, that if Δ is integrable and $X, Y \in \Delta$, then $[X, Y] \in \Delta$. It turns out the converse is also true (at least locally): if for every vector fields $X, Y \in \Delta$, we have $[X, Y] \in \Delta$, then Δ is (locally) integrable.

Let θ^α, $\alpha = 1, \ldots, m-k$ be a set of linearly independent 1-forms that annihilate Δ. Show that the following four conditions are equivalent:

(i) For every $X, Y \in \Delta$, $[X, Y] \in \Delta$.

(ii) $d\theta^\alpha(X, Y) = 0$ for all vector fields $X, Y \in \Delta$.

(iii) There exist 1-forms ${\lambda^\alpha}_\beta$ such that $d\theta^\alpha = {\lambda^\alpha}_\beta \wedge \theta^\beta$.

(iv) $d\theta^\alpha \wedge \Omega = 0$, where $\Omega = \theta^1 \wedge \ldots \wedge \theta^{m-k}$.