Physics 250
Fall 2008
Homework 13
Due Monday, December 1, 2008

Reading Assignment: Nakahara, pp. 293–296; pp. 302–307, if you're interested in string theory; pp. 348–373. See also Frankel, chapters 16 and 17.

Notes. The "highly technical" proof alluded to in Exercise 7.23, p. 295, is highly technical because of the machinery of functional analysis needed to make precise statements. If you proceed with the usual standards of rigor in quantum physics, it's not at all hard. You just assume the Laplacian has a complete set of eigenfunctions (i.e., eigenforms) (because it is Hermitian) and that the spectrum is discrete (because M is compact). Then the equation $\triangle\omega = \psi$ (ψ is given, ω is unknown, it is a generalized Poisson equation) can be solved for ω if and only if ψ is orthogonal to the space of harmonic forms. You see this by expanding both ω and ψ in the eigenbasis of $\triangle$.

I refrained from using the eigenbasis of $\triangle$ in my presentation in lecture, but several of the theorems take on a revealing form if you expand everything in this eigenbasis. If M is not compact, then $\triangle$ has a continuous spectrum, and you need to use Green's functions. Green's functions can also be used in the case that the metric is not positive definite, although then you get into issues such as forward and retarded solutions. In that case $\triangle$ is a generalized d'Alembertian operator (a wave operator).

In lecture I did (or will) cover most of the material in Nakahara's Chapter 9, but I tried to make it more clear and motivated. So you might find it most useful to read the lecture notes first, then Nakahara's Chapter 9. This chapter does what can be done with fiber bundles before you introduce a connection, which is the subject of Chapter 10.

On p. 351, Nakahara requires the transition functions t_{ij} to satisfy conditions (9.6a–c), but these conditions follow immediately from the definitions of the t_{ij} (his Eq. (9.4)). (You do not have to impose any extra requirements.)

On p. 352, below Eq. (9.8), Nakahara says that the g_i should be homeomorphisms. For all our applications, they should be diffeomorphisms. In the following paragraph, I think Nakahara means $\Gamma(M, E)$ for the set of sections over M (he has F instead of E).

On the last line of p. 353, Nakahara writes $f' = t_{ij} f$, but he should swap f and f'. This error was also in the first edition.

I'll make more remarks about Nakahara's chapter 9 next week.

1. (DTB) Some easy problems.

(a) Show that $*\alpha$ is coclosed iff α is closed. Show that $*\alpha$ is coexact iff α is exact.

(b) On a compact, orientable Riemannian manifold without boundary, prove Poincaré duality, i.e., $b_r(M) = b_{m-r}(M)$. Hint: Don't try to follow Nakahara's logic on Poincaré duality, use the theory of harmonic forms. The answer doesn't depend on the metric.

2. A problem on fiber bundles. As pointed out in class, if H is a Lie subgroup of a Lie group G, then the foliation of G into left cosets of H gives G the structure of a principal fiber bundle, in which the entire space $P = E$ is G, the standard fiber F is H, and the base manifold is the space of left cosets, $M = G/H$. Here the fibers (the left cosets) are the orbits of the right action of H on G, $h \mapsto R_h$, and the projection is $\pi : G \to G/H : g \mapsto [g] = gH$.

Suppose $G = SO(3)$ and H is the $SO(2)$ subgroup consisting of rotations about the z-axis. It was proved in an earlier homework (and in lecture) that $G/H = S^2$. Find an open cover and local trivializations, that is, the set $\{(U_i, \phi_i)\}$ as explained in the notes, where $\phi_i : U_i \times F \to \pi^{-1}(U_i)$. Here $\{U_i\}$ is an open cover of S^2. Then find the transition functions $t_{ij} : U_i \cap U_j \to H$.

Hint: Use the formula,

$$R(\hat{\mathbf{a}}, \alpha) R(\hat{\mathbf{b}}, \beta) R(\hat{\mathbf{a}}, -\alpha) = R(\hat{\mathbf{b}}', \beta), \tag{1}$$

where $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are two axes and α and β are two angles, and where

$$\hat{\mathbf{b}}' = R(\hat{\mathbf{a}}, \alpha)\hat{\mathbf{b}}. \tag{2}$$

Also note that each ϕ_i must be a *smooth* map, for example, if U_i includes the north pole, then ϕ_i must approach a value as you approach the north pole that is independent of the direction from which you approach it.