Physics 250
Fall 2008
Homework 12
Due Friday, November 21, 2008

Reading Assignment: Nakahara, pp. 289–293. For more on Hodge star theory and harmonic forms, see Frankel, pp. 361–374. As usual, Frankel gives a good supplement to the material in Nakahara.

Notes. Regarding Nakahara's use of a noncoordinate basis in Sec. 7.9, he usually means an orthonormal basis when he talks about a noncoordinate basis, whereas in class I have usually used the symbols $\{\theta^\mu\}$ to stand for any basis, coordinate or noncoordinate, orthonormal or not, because almost everything in this section goes through without modification in the general case.

On p. 290, Nakahara calls ϵ a "tensor," but as I showed in class, it does not transform as a tensor. For that reason, in class I refrained from raising indices on ϵ, creating things like Nakahara's Eq. (7.171b). Instead, I used Ω (which is a tensor).

The following are useful identities when dealing with the permutation (or Levi-Civita) symbol ϵ. First,

$$\epsilon_{\mu_1\ldots\mu_r\nu_1\ldots\nu_{m-r}}\,\epsilon_{\sigma_1\ldots\sigma_r\nu_1\ldots\nu_{m-r}} = (m-r)!\,\mathrm{sgn}\begin{pmatrix}\mu_1\ldots\mu_r\\ \sigma_1\ldots\sigma_r\end{pmatrix}, \tag{1}$$

where sgn() means ± 1 if the first row of integers is an even/odd permutation of the bottom row, and 0 otherwise. This is my notation, no one else uses it as far as I know. The $(m-r)!$ occurs because we have a contraction between two sets of indices, here the ν's, in which the objects contracted are completely antisymmetric. The sgn() notation can be written in terms of a matrix of Kronecker δ's,

$$\mathrm{sgn}\begin{pmatrix}\mu_1\ldots\mu_r\\ \sigma_1\ldots\sigma_r\end{pmatrix} = \begin{vmatrix}\delta^{\mu_1}_{\sigma_1} & \ldots & \delta^{\mu_1}_{\sigma_r}\\ \vdots & & \vdots\\ \delta^{\mu_r}_{\sigma_1} & \ldots & \delta^{\mu_r}_{\sigma_r}\end{vmatrix}. \tag{2}$$

This notation is a generalization of the ϵ symbol, since

$$\epsilon_{\mu_1\ldots\mu_r} = \mathrm{sgn}\begin{pmatrix}12\ldots r\\ \mu_1\mu_2\ldots\mu_r\end{pmatrix}. \tag{3}$$

In many cases sgn() behaves like a big Kronecker δ, for example,

$$\mathrm{sgn}\begin{pmatrix}\alpha\beta\gamma\\ \mu\nu\sigma\end{pmatrix}\,\theta^\mu\wedge\theta^\nu\wedge\theta^\sigma = 3!\,\theta^\alpha\wedge\theta^\beta\wedge\theta^\gamma. \tag{4}$$

This is because the sgn() part of the expression vanishes unless $(\mu\nu\sigma)$ is a permutation of $(\alpha\beta\gamma)$, and there are 3! such permutations, each of which gives the same answer. So we can just choose one of these permutations and multiply the answer by 3!. The easiest one is to choose $\mu=\alpha$, $\nu=\beta$, $\sigma=\gamma$.

I used the symbol $\langle\, ,\, \rangle$ for the scalar product of forms instead of $(\, ,\,)$, since the rounded brackets were used earlier for pairing a form with a chain (a vector and a dual vector, instead of two vectors). Thus, in my notation, $(\, ,\,)$ does not require a metric, while $\langle\, ,\, \rangle$ does.

Notice that when Nakahara computes the covariant Laplacian (actually, the negative of the Laplacian in usual physics parlance) on p. 294, he uses the Levi-Civita connection in deriving Eq. (7.188). Hodge star theory and the definition of $d^\dagger$ use a metric, but not a connection.

1. (DTB) Work in curved, four-dimensional space-time. In class we showed that the covariant derivative of a Dirac spinor was defined by

$$\nabla_\gamma \psi = \psi_{,\gamma} - \frac{i}{4} \Gamma^\alpha_{\gamma\beta}\, \sigma_\alpha{}^\beta\, \psi. \tag{5}$$

Here as in the book Greek indices α, β, γ etc. at the beginning of the Greek alphabet (vierbein indices) refer to components with respect to an orthonormal vierbein $\{e_\alpha\}$, and Greek indices in the middle of the Greek alphabet, $\mu, \nu, \lambda, \kappa$ etc. (coordinate indices) refer to coordinates x^μ in some coordinate system (although in lecture I didn't follow this convention). The vierbein is specified by

$$e_\alpha = e_\alpha{}^\mu(x)\, \frac{\partial}{\partial x^\mu}, \tag{6}$$

where

$$g(e_\alpha, e_\beta) = \eta_{\alpha\beta}, \tag{7}$$

where $\eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric. All vierbein indices are raised and lowered with $\eta_{\alpha\beta}$. The comma notation when used with vierbein indices, as in Eq. (5), means, for example,

$$\psi_{,\gamma} = e_\gamma \psi = e_\gamma{}^\mu\, \psi_{,\mu} = e_\gamma{}^\mu\, \frac{\partial \psi}{\partial x^\mu}. \tag{8}$$

A gauge transformation in general relativity is a local Lorentz transformation on the vierbein,

$$e'_\alpha = \Lambda_\alpha{}^\beta\, e_\beta, \tag{9}$$

where $\Lambda^\alpha{}_\beta$ is the matrix of a Lorentz transformation as in special relativity,

$$\Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta\, \eta_{\alpha\beta} = \eta_{\gamma\delta}, \tag{10}$$

or,

$$(\Lambda^{-1})^\alpha{}_\beta = \Lambda_\beta{}^\alpha. \tag{11}$$

Note that $\Lambda_\alpha{}^\beta$ in Eq. (9) depends on x.

Use the following conventions for the transformation of a Dirac spinor under Lorentz transformations in special relativity (don't try to follow Nakahara, I think there are errors in his presentation). There is a wide variety of conventions used in the literature for the formalism of the Dirac equation, but I think the ones I use here are the most common in physics (they are essentially those of Bjorken and Drell). Nakahara uses some non-standard conventions.

A contravariant vector transforms according to

$$X'^{\alpha} = \Lambda(g)^{\alpha}{}_{\beta}\, X^{\beta}, \tag{12}$$

where $g \in SL(2,\mathbb{C})$, and the spinor transforms according to

$$\psi' = D(g)\psi, \tag{13}$$

where $D(g)$ is a 4×4 spinor (Dirac) representation of the proper orthochronous Lorentz group. This representation has the properties,

$$D(g_1)D(g_2) = D(g_1 g_2), \tag{14}$$

$$D(g)^{-1}\,\gamma^{\alpha} D(g) = \Lambda(g)^{\alpha}{}_{\beta}\,\gamma^{\beta}, \tag{15}$$

and

$$\gamma^0 D(g)^{\dagger}\gamma^0 = D(g)^{-1}. \tag{16}$$

Here γ^{α} are the usual Dirac matrices, which satisfy the anticommutation relations,

$$\{\gamma^{\alpha},\gamma^{\beta}\} = 2\eta^{\alpha\beta}. \tag{17}$$

The relation between infinitesimal Lorentz transformations and infinitesimal spinor transformations is the following. If an infinitesimal Lorentz transformation is written in the form,

$$\Lambda(g)^{\alpha}{}_{\beta} = \delta^{\alpha}_{\beta} + \epsilon\Omega^{\alpha}{}_{\beta}, \tag{18}$$

where ϵ is just a reminder that the correction is small and where

$$\Omega_{\alpha\beta} = -\Omega_{\beta\alpha}, \tag{19}$$

then

$$D(g) = 1 - \frac{i}{4}\epsilon\Omega_{\alpha\beta}\,\sigma^{\alpha\beta}, \tag{20}$$

where

$$\sigma^{\alpha\beta} = \frac{i}{2}[\gamma^{\alpha},\gamma^{\beta}]. \tag{21}$$

Show explicitly that $\nabla_{\alpha}\psi$ transforms as a spinor (in its Dirac indices) and as a covector (in the index α).

2. (DTB) Let $A = A_{\mu}\,\theta^{\mu}$ be a 1-form on a manifold with a metric g. It was shown in class that

$$d^{\dagger}A = -A^{\mu}{}_{;\mu}. \tag{22}$$

In this problem we use the Levi-Civita connection.

(a) As discussed in class, the inhomogeneous Maxwell equation in general relativity (with a 4-dimensional, pseudo-Riemannian manifold) is

$$F^{\mu\nu}{}_{;\nu} = J^{\mu}, \tag{23}$$

where we set $c = 1$ and use Heaviside-Lorentz units (which get rid of the 4π's). It was reported in class that this equation is equivalent to

$$d^\dagger F = J, \tag{24}$$

where J is the current 1-form,

$$J = J_\mu \, dx^\mu. \tag{25}$$

Let

$$B = \frac{1}{2} B_{\mu\nu} \, \theta^\mu \wedge \theta^\nu \tag{26}$$

be an arbitrary 2-form on an arbitrary manifold with a metric g. Compute $d^\dagger B$ in terms of the components $B_{\mu\nu}$. Use only covariant derivatives, as in Eq. (22) above, to make it obvious that the answer is a tensor. Once you have your answer, specialize to the case $B = F$ to prove Eq. (24).

Note, based on the quoted answer (24) above, you might guess that

$$d^\dagger B = B_\mu{}^\nu{}_{;\nu} \, \theta^\mu, \tag{27}$$

but remember that $dF = 0$ while B is arbitrary, so don't jump to conclusions.

(b) In class we showed that if f is a scalar, then

$$\triangle f = -f^{;\mu}{}_{;\mu}. \tag{28}$$

If $A = A_\mu \, \theta^\mu$ is a 1-form, we might guess that

$$\triangle A = -A_\mu{}^{;\nu}{}_{;\nu} \, \theta^\mu. \tag{29}$$

Work out $\triangle A$ in terms of components, write the answer purely in terms of covariant derivatives, and see if the guess is right.