Physics 250
Fall 2008
Homework 11
Due Friday, November 14, 2008

Reading Assignment: Nakahara, pp. 273–289, 297–302.

Notes. On pp. 275–277, Nakahara derives the expression for the Weyl tensor, following the "elegant" coordinate-free approach of Nomizu. This derivation includes "straightforward but tedious" calculations. I found it straightforward and not so tedious just to do it in coordinates. Here are my main results. Write $\bar{g}_\mu = e^{2\sigma}\, g_{\mu\nu}$. Then you find,

$$\bar{\Gamma}^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} + \delta^\mu_\alpha\, \sigma_{,\beta} + \delta^\mu_\beta\, \sigma_{,\alpha} - g_{\alpha\beta}\, g^{\mu\tau}\, \sigma_{,\tau}. \tag{1}$$

Then define

$$B_{\mu\nu} = \sigma_{;\mu\nu} - \sigma_{,\mu}\sigma_{,\nu} + \tfrac{1}{2} g_{\mu\nu}(g^{\alpha\beta}\sigma_{,\alpha}\, \sigma_{,\beta}). \tag{2}$$

Then calculate the Riemann tensor, and you find,

$$\bar{R}_{\mu\nu\alpha\beta} = e^{2\sigma}(R_{\mu\nu\alpha\beta} + g_{\mu\beta}\, B_{\nu\alpha} - g_{\mu\alpha}\, B_{\nu\beta} + g_{\alpha\nu}\, B_{\mu\beta} - g_{\beta\nu} B_{\mu\alpha}). \tag{3}$$

Taking traces, you find

$$\bar{R}_{\mu\nu} = R_{\mu\nu} - (m-2)B_{\mu\nu} - g_{\mu\nu}\, B^\alpha{}_\alpha, \tag{4}$$

and

$$\bar{R} = e^{-2\sigma}[R - 2(m-1)B^\alpha{}_\alpha], \tag{5}$$

where $m = \dim M$. Then substitute back, eliminate $B_{\mu\nu}$ in favor of $\bar{R}_{\mu\nu}$ and $R_{\mu\nu}$. This brings in $\operatorname{tr} B = B^\alpha{}_\alpha$, and you eliminate that using Eqs. (4) and (5). You get an expression involving barred and unbarred tensors, which can be put into the form,

$$\bar{W}^\mu{}_{\nu\alpha\beta} = W^\mu{}_{\nu\alpha\beta}, \tag{6}$$

where

$$\begin{aligned} W_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} &+ \frac{1}{m-2}(g_{\mu\beta}\, R_{\alpha\nu} - g_{\mu\alpha}\, R_{\beta\nu} + g_{\alpha\nu}\, R_{\mu\beta} - g_{\beta\nu}\, R_{\mu\alpha}) \\ &+ \frac{1}{(m-1)(m-2)}(g_{\mu\alpha}\, g_{\beta\nu} - g_{\mu\beta}\, g_{\alpha\nu})R. \end{aligned} \tag{7}$$

On p. 278, Nakahara gives the proof that any 2-dimensional Riemannian manifold is conformally flat, which was skipped in lecture.

On p. 283, Nakahara uses the convention of using letters at the beginning of the Greek alphabet, α, β, etc., for vielbein indices, and letters in the middle of the Greek alphabet, μ, ν, κ, etc., for coordinate indices. I did not follow this convention in lecture, where the type of basis used (coordinate or noncoordinate) was determined by context. As mentioned in class, a *vierbein* is also called a *tetrad* (with *dreibein*=*triad*, etc). Also, a coodinate (noncoordinate) basis is sometimes called a *holonomic* (*anholonomic*) basis. Also, Nakahara seems to restrict the use of the terminology *vielbein* to the case of an orthonormal basis, in which $g(e_\alpha, e_\beta) = \delta_{\alpha\beta}$ or $\eta_{\alpha\beta}$, but most of the formalism works fine for any basis (coordinate or noncoordinate, orthogonal or not), as pointed out in lecture. In particular, Cartan's structure equations and his generalized versions of the Bianchi identities do not require an orthonormal basis (in fact, they do not even require a metric). BTW, in case you have not seen the notation before, $\eta_{\mu\nu}$ stands for the metric of special relativity, $\mathrm{diag}(+1, -1, -1, -1)$, that is, it is the Minkowski equivalent of $\delta_{\alpha\beta}$ (on a Riemannian manifold).

We temporarily skipped Hodge star theory and harmonic forms, pp. 289–296, but will do them next week.

Nakahara writes *Ric* for the Ricci tensor, whereas I have written just $R_{\mu\nu}$. There is no confusion as long as you attach indices (the number of indices distinguishes the Ricci tensor from the curvature tensor $R_{\mu\nu\alpha\beta}$), but if you want to use more abstract (index-free) notation then you do need his notation.

I followed Nakahara pretty closely in lecture for the derivation of the Einstein field equations from a variational principle, but I am taking a different (and I think much simpler) approach to spinors in curved space time, which Nakahara covers in section 7.10.3, pp. 300–302. His discussion is quite complicated, and I don't think he follows standard conventions for γ matrices, etc. The general relativistic aspects of this problem are quite simple, as will be shown in class. The only complication is due to the details of the Dirac (4-spinor) representation of the Lorentz group, which will be remarked upon in class. I advise you to skip this part of the reading, and just read the lecture notes.

1. Consider the Poincaré half-plane, with metric

$$g = \frac{dx^2 + dy^2}{y^2}. \tag{8}$$

See pp. 265–266, and watch out for the error noted on last week's homework. We only use the region $y > 0$. The geodesics in this metric are calculated in the book.

(a) Using the methods of Sec. 7.8.4, compute the curvature 2-form, $R^\alpha{}_\beta$. The Poincaré half-plane is a surface of constant negative curvature.

(b) Find the Killing vector fields for this metric. Do this any way you like, but I found them by writing the Killing vector field in the form,

$$X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y}, \tag{9}$$

and then finding a differential equation for X and Y.

(c) Show that the Killing vectors form a Lie algebra.

One can show that the advance maps generated by the Killing vectors for this problem can be expressed as fractional linear transformations,

$$z' = \frac{az+b}{cz+d}, \tag{10}$$

where $z = x + iy$. The identity component of the isometry group is $SO(2,1)$.

2. Nakahara problem 7.2, p. 307 (problem 2, p. 264 of the first edition).