Physics 250
Fall 2008
Homework 10
Due Friday, November 7, 2008

Reading Assignment: Nakahara, pp. 251–273. See also Frankel, pp. 232–246.

Notes. Just a few comments. On pp. 258–259, Nakahara is trying to point out that if the connection coefficients $\Gamma^\mu_{\alpha\beta}$ vanish in some coordinate system, then it means that the rule of parallel transport (in those coordinates) is that the parallel transported vector has the same components as the original vector. This applies, for example, to parallel transport in a vector space, using linear coordinates. One can similarly define such a (trivial) connection on any parallelizable manifold, which includes group manifolds, as pointed out in class. Obviously, if the connection coefficients vanish (in some coordinates), then the curvature tensor $R^\mu{}_{\nu\alpha\beta}$ vanishes (in all coordinates).

On p. 253, Nakahara says that the variational condition gives the local extremum of the length of a curve between two points. Actually, it is only a stationary point, in general (a kind of a saddle point in function space).

On p. 265, Nakahara's equation (7.68b) is wrong, but he never uses it in the subsequent derivation. Just below that, I can't see what dividing by x' has to do with anything. Actually, the derivation of the geodesic equations are much easier if you just use a Lagrangian. In the present case, let

$$L(x,y,x',y') = \frac{1}{2}\frac{x'^2+y'^2}{y^2}, \tag{10.1}$$

where the 1/2 is only for convenience. (Also, we could have used the square root of the above expression, but the answers will be the same and the above expression is easier to work with.) You will find the Euler-Lagrange equations give you the geodesic equations immediately, and also (by Noether's theorem on the ignorable coordinate x) they give you the integral (7.69).

Note: This homework has 5 problems, but you only need to do 4 to get 100% credit. You can choose which 4 you want to do.

1. (DTB) Let M be a submanifold of Euclidean $\mathbb{R}^n$. The metric on $\mathbb{R}^n$ is

$$g = \sum_{i=1}^n dx^i \otimes dx^i, \tag{10.2}$$

where $\{x^i, i=1,\ldots,n\}$ are the standard coordinates on $\mathbb{R}^n$. Let $\{x^\mu, \mu=1,\ldots,m\}$ (with $m \le n$) be coordinates on M, which is specified by functions $x^i = x^i(x^\mu)$. Let the metric on the submanifold be the metric on $\mathbb{R}^n$, restricted to the submanifold. The metric on M has components $g_{\mu\nu}$. Let x^μ and $x^\mu + \xi^\mu$ be coordinates of two nearby points (call them x and $x+\xi$) on M (ξ^μ is infinitesimal).

As discussed in class, we define a connection on M as follows. We take a tangent vector X in T_xM, reinterpret it as a tangent vector in $T_x\mathbb{R}^n$, parallel transport it over to $T_{x+\xi}\mathbb{R}^n$ by using the vector space structure of $\mathbb{R}^n$, then project it onto $T_{x+\xi}M$ using the metric in $\mathbb{R}^n$. Find the connection coefficients $\Gamma^\mu_{\alpha\beta}$ in terms of $g_{\mu\nu}$.

2. (DTB) Given (M, ∇, g) (manifold with connection plus metric), and assume $\nabla g = 0$, but don't make any other assumptions. In particular, don't assume that the torsion $T = 0$. Prove that

$$g(W, R(X, Y)Z) + g(R(X, Y)W, Z) = 0, \tag{10.3}$$

where X, Y, Z, W are vector fields. Do this in coordinate-free notation. What symmetry does this imply for $R^\mu{}_{\nu\alpha\beta}$?

3. Consider an accelerated particle moving in Minkowski space-time, with metric

$$g = dt^2 - dx^2 - dy^2 - dz^2. \tag{10.4}$$

The particle's world line is $x^\mu(\tau)$, where τ is proper time, and its spin 4-vector is s^μ. The spin is required to be a purely spatial vector in the particle's rest frame, so $s^\mu u_\mu = 0$, where $u^\mu = dx^\mu/d\tau$. In the absence of torques (e.g., magnetic fields) on the spin, the spin vector evolves according to the equation of Fermi-Walker transport,

$$\frac{ds^\mu}{d\tau} = -u^\mu(s^\nu a_\nu), \tag{10.4}$$

where $a^\mu = du^\mu/d\tau$. This equation guarantees that s^μ is orthogonal to u^μ for all τ if it is orthogonal at $\tau = 0$ (as we assume).

If the velocity of the particle undergoes a cycle in some time interval, so that the final velocity is equal to the initial velocity, then the final 3-dimensional space-like hyperplane that the spin lies in is the same at the beginning and ending of the the time interval (say, τ_0 to τ_1). (The particle might be undergoing periodic motion.) Under these circumstances, it is meaningful to talk about the 3-dimensional rotation (in the 3-dimensional, space-like hyperplane) that maps the initial spin into the final spin. This rotation is usually called Thomas precession.

The world velocity of the particle u^μ satisfies

$$u^\mu u_\mu = 1. \tag{10.4}$$

Thus, the vector u^μ lies on a hyperboloid in "world velocity space". Call the hyperboloid H. "World velocity space" has the same metric $g_{\mu\nu}$ as Minkowski space. We create a map from the world line of the particle to H by means of the function $u^\mu(\tau)$, which defines a curve on H. Call this the "H-map." If the velocity is cyclic, as assumed, then the curve on H forms a loop with a base point $u^\mu(\tau_0) = u^\mu(\tau_1)$.

As pointed out in class, the surface H is actually a purely space-like hypersurface of constant negative curvature, if its metric is taken to be the (Minkowski) metric of the imbedding space, restricted to H. Explain why the rotation of Thomas precession is the holonomy of the loop on H, with respect to this metric on H.

In this problem we are dealing with what is called the "normal bundle", that is, the set of vectors normal to a submanifold of a (pseudo)-Riemannian manifold. In this case, the submanifold is the 1-dimensional world line of the particle. The rate of rotation of the spin along the world line is the angular velocity of Thomas precession, essentially an $\mathfrak{so}(3)$-valued 1-form on the world line, which is the pull-back of the Levi-Civita connection on H to the world line by means of the H-map.

4. (DTB) As explained in class, the phase space of a classical system is a *symplectic manifold*, one upon which there exists a nondegenerate, closed 2-form ω (that is, satisfying $d\omega = 0$ and $\det \omega_{\mu\nu} = 0$). In the subject known as "deformation quantization," the idea is to deform the usual (pointwise, commutative) multiplcation law for classical observables, by introducing an $\hbar$ dependence into the multiplication law. When $\hbar = 0$, we have the classical algebra of observables, but when $\hbar$ is switched on, the multiplication law becomes noncommutative (but it is still associative). The new multiplication law is interpreted as that of quantum observables.

In deformation quantization, it is necessary to introduce a "symplectic connection," that is, a connection ∇ that preserves the symplectic 2-form, $\nabla\omega = 0$. It is usually assumed that this connection has vanishing torsion. In the analogous circumstances in metrical geometry, it is possible to solve for the connection coefficients $\Gamma^{\mu}_{\alpha\beta}$ in terms of the metric tensor $g_{\mu\nu}$ and its derivatives (this is the Levi-Civita connection). Can this also be done for a symplectic connection? Show that if Γ and $\bar{\Gamma}$ are the components of two torsion-free symplectic connections, then the difference

$$\Gamma_{\mu\alpha\beta} - \bar{\Gamma}_{\mu\alpha\beta} \tag{10.7}$$

is a completely symmetric tensor (here $\Gamma_{\mu\alpha\beta} = \omega_{\mu\nu}\Gamma^{\nu}_{\alpha\beta}$).

5. (DTB) Let ω be a 2-form on a manifold with a connection ∇, and let X, Y and Z be vector fields. Express

$$(\nabla_X\omega)(Y,Z) + (\nabla_Y\omega)(Z,X) + (\nabla Z\omega)(X,Y) \tag{10.8}$$

in terms of $d\omega(X,Y,Z)$. What happens when the torsion vanishes? Notice the similarity with the second Bianchi identity.