Physics 250
Fall 2008
Homework and Notes 1
Due Friday, September 5, 2008

About Homework: I will try to have weekly homework, but there may be some weeks without it. Nakahara's problems are usually not very good, so I will try to do better. The homework will be made available on the web site by Friday of each week, and will be due at 5pm on Friday of the following week, in the envelope hanging outside my office (449 Birge).

In any homework exercise marked "DTB" (meaning, "Done This Before"), you may get full credit by simply stating, "DTB". Please also say where you have done it before, such as, "Math 799 at Appalachian State Teacher's College" or "Self Study".

Reading Assignment: Lecture notes for 8/28/08, Nakahara, Chapter 1. Also please read into Chapter 2 of Nakahara to cover approximately the same material as in the lecture notes. I won't quote page numbers because at this point I don't have a copy of Nakahara's 2nd edition, but I'll try to do that in the future.

1. (DTB) A common way to obtain equivalence classes is through a group action. Let G be a group and M a space. A *group action* is an association between elements $g \in G$ and bijections $\Phi_g : M \to M$, such that $\Phi_g \Phi_h = \Phi_{gh}$. Properly speaking, the action itself is the mapping : $g \mapsto \Phi_g$.

Note that the space of bijections of M onto itself is itself a group, with composition being the multiplication law. Thus, a group action can be regarded as a group homomorphism between G and this space of bijections. In many physical applications, M is a differentiable manifold and the maps Φ_g are diffeomorphisms (terminology to be explained later). The maps Φ_g are then transformations of M onto itself (rotations, Lorentz transformations, canonical tranformations, etc).

(a) Given a group action on a space M, we may consider two points $x, y \in M$ as equivalent if $y = \Phi_g x$ for some $g \in G$. Show that this is an equivalence relation. The equivalence class $[x]$ is called the *orbit* of x under the group action; for example, think of the spheres (S^2) which result by applying rotations $SO(3)$ to a point of $\mathbb{R}^3$. Thus, the space M is broken up into disjoint subsets, the orbits of the group action.

(b) Given a group G and a subgroup H, the *left cosets* of H are the sets $[g] = \{gh | h \in H\}$, where g is the representative element of the coset. Similarly, the *right coset* of H containing g is the set $\{hg | h \in H\}$. Show that the left and right cosets are orbits of (two different) group actions of H on G (identify the respective group actions). (Left and right cosets are sometimes denoted gH and Hg, respectively.)

(c) In Nakahara's exercises 2.6 and 2.7, he wants you to show that the relation defined is an equivalence relation by appealing to the definition of an equivalence relation. Do these problems instead by showing that equivalent points lie on the orbit of some group action. In problem 2.6, $G = SL(2,\mathbb{Z})$ and $M = H$, and in problem 2.7, G is any group that acts on itself ($M = G$) by conjugation.

Here is a confusing point regarding the terminology of group actions. The group action defined above is sometimes called a "left action." In this course, all group actions will be left actions, so we'll omit the "left". But for reference, here is the definition of a "right action." A right action of a group G on a space M is an association between elements $g \in G$ and bijections Φ_g of M onto itself such that $\Phi_g \Phi_h = \Phi_{hg}$ for all $g, h \in G$ (the Φ products are in the reverse order from a left action). Any right action is closely associated with a left action (this is why we will only use left actions in this course). To see this, suppose we have a right action : $g \mapsto \Phi_g$. Then define a different mapping between the group and the same set of bijections Φ_g of M onto itself by $\Psi_g = \Phi_{g^{-1}}$. Then

$$\Psi_g \Psi_h = \Phi_{g^{-1}} \Phi_{h^{-1}} = \Phi_{h^{-1}g^{-1}} = \Phi_{(gh)^{-1}} = \Psi_{gh}. \tag{1}$$

Thus, the map : $g \mapsto \Psi_g$ is a left action.

2. Let $SO(3)$ be the usual group of proper rotations, with the usual action on $\mathbb{R}^3$. Let $SO(2)$ be the subgroup of rotations about the z-axis. Let $R \in SO(3)$ and write R in Euler angle form,

$$R = R_z(\alpha) R_y(\beta) R_z(\gamma). \tag{2}$$

Consider the left cosets of $SO(2)$ within $SO(3)$. If two rotations belong to the same left coset, what can you say about their Euler angles? What about right cosets?

Find the topology of the quotient space, $SO(3)/SO(2)$ (use left cosets). Hint: Don't try to do this in Euler angles, they are ugly. Instead, consider the action of an arbitrary rotation in $SO(3)$ on the z-axis, and use it to characterize the cosets.

3. The Hopf fibration. Consider S^3 as the space of normalized, 2-component spinors, that is, as the unit sphere in $\mathbb{C}^2 = \mathbb{R}^4$. Define an equivalence relation on S^3 by setting $x \sim y$ if $x = e^{i\alpha} y$, where x and y are viewed as unit vectors in $\mathbb{C}^2$ or $\mathbb{R}^4$. As discussed in class, the equivalence classes are circles. Show that the quotient space $S^3/\sim$ can be placed in one-to-one correspondence with points of S^2, that is, show that

$$\frac{S^3}{S^1} = S^2. \tag{3}$$

(In this notation, we use S^1, the equivalence class, in the denominator, instead of $\sim$, the equivalence relation. The notation requires some context to be understood completely.) You will find hints for how to do this problem if you read the lecture notes.