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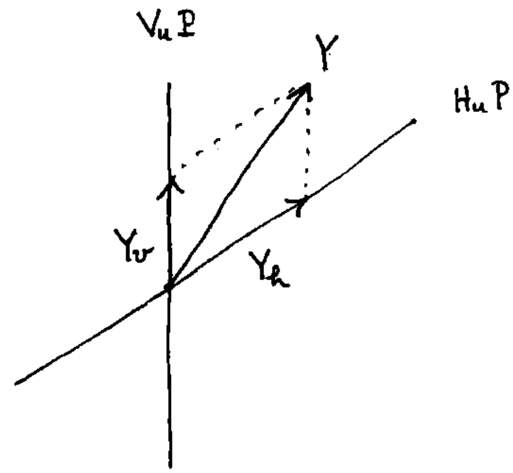
We now develop an alternative way to define a connection. It is related to the idea that a vector subspace (like H_uP) can be specified by the forms that annihilate it. Let us sketch T_uP, with horiz. and vertical subspaces: An arbitrary vector $Y \in T_uP$ can be uniquely decomposed into horiz. and vert. components,



$$Y = Y_v + Y_h$$

Define projection operators Π_h, Π_v such that

$$Y_v = \Pi_v(Y),$$

$$Y_h = \Pi_h(Y),$$

$$\Pi_v + \Pi_h = 1.$$

The vertical projector is a map $\Pi_v : T_uP \to V_uP$. It has the property that $H_uP = \ker \Pi_v$.

The vertical space V_uP is isomorphic with the Lie algebra $\mathfrak{g}$ of G. This is true for all u (a different isomorphism for each u). This comes about because of the (right) action of G on P, $a \mapsto R_a$, $a \in G$, $R_a : P \to P$. Actually, since R_a preserves fibers, we can think $R_a : F_x \to F_x$. This action induces vector fields on F_x, which are purely vertical (as seen within P).

In the following it is convenient to denote a vector at a point as an equivalence class of curves, say, $X = [\sigma(t)]$ where $\sigma(0) = x$

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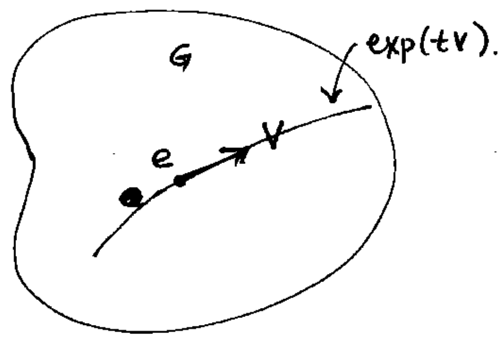
and $X \in T_x M$. This notation makes it convenient to apply the tangent map f_* to the vector (where say $f : M \to N$), according to

$$f_*[\sigma(t)] = [f\sigma(t)].$$

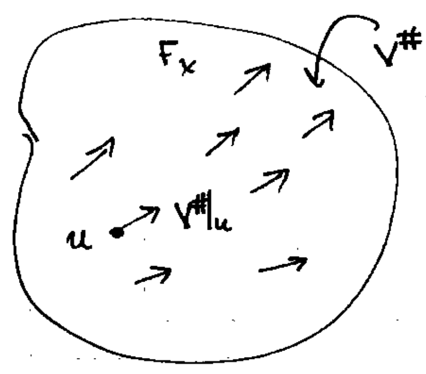
It is a convenient way of dealing with tangent maps in coordinate-free notation.

For example, let $V \in \mathfrak{g}$. A curve passing through e at $t=0$ with tangent vector V at $t=0$ is $\exp(tV)$ (the integral curve of the left- or right-invariant vector field ~~evaluated~~ associated with V). That is,



$$V = [\exp(tV)].$$

Letting G act on F_x by the right action R_a, we get an induced vector field $V^\#$ on F_x, corresponding to $V \in \mathfrak{g}$. Note that V is just on vector, while $V^\#$ is a vector field.
$V^\#$ evaluated at a point $u \in F_x$ can be seen as an equivalence class of curves,



$$V^\#|_u = [R_{\exp(tV)}\, u] = [u \exp(tV)].$$

This is the definition of $V^\#$ (equivalent to the one given earlier, but note that earlier we used left actions instead of right actions.)

Using this construction we get a map: $\mathfrak{g} \to T_u(F_x) : V \mapsto V^\#|_u$.

This map is invertible (it is a vector space isomorphism) because the right action of R_a on F_x is <u>free</u>. We can call this map # or #$|_0$.

To go back to the vertical projector Π_v, we can construct a chain of maps,

$$T_u P \xrightarrow{\Pi_v} V_u P \xrightarrow{\#^{-1}} \mathfrak{g}.$$

Composing these, we get the definition of a Lie algebra-valued 1-form on P, called the <u>Ehresmann form</u>:

<u>Def</u>: $\omega = \#^{-1} \circ \Pi_v$

$$\omega|_u : T_u P \to \mathfrak{g}$$

$$\omega \in \mathfrak{g} \otimes \Omega^1(P).$$

The Ehresmann form has the following properties. First, it annihilates the horizontal subspace, because Π_v does so:

$$\omega|_u (H_u P) = 0.$$

Second, if we let ω act on an arbitrary ~~and~~ vertical vector, write it $V^\#|_u$ for some $V \in \mathfrak{g}$, then the Π_v does nothing and $\#^{-1}$ strips of the #:

$$\omega|_u (V^\#|_u) = V, \qquad \forall \quad V \in \mathfrak{g}.$$

Third, ω has a certain behavior when pulled back by R_a. Consider

$$(R_a^* \omega)|_u (V^\#|_u) = \omega|_{ua} (R_{a*} (V^\#|_u)),$$

where we evaluate on an arbitrary vertical vector and use the defn. of the pull-back. Now, from above we have

$$R_{a*}(V^{\#}|_u) = R_{a*}\,[u \exp(tV)]$$
$$= [R_a\, \cancel{\exp}\; u \exp(tV)]$$
$$= [u \exp(tV)\, a]$$
$$= [u a\, a^{-1} \exp(tV)\, a].$$

Now we use the identity,

$$a^{-1} \exp(tV) a = \exp(t\, Ad_{a^{-1}} V),$$

(more on this identity below).

where $g \mapsto Ad_g$ is the adjoint representation of G. Thus,

$$= [ua \exp(t\, Ad_{a^{-1}} V)]$$
$$= (Ad_{a^{-1}} V)^{\#}|_{ua}.$$

So,

$$(R_a^* \omega)|_u\,(V^{\#}|_u) = \omega|_{ua}\left((Ad_{a^{-1}} V)^{\#}|_{ua}\right)$$
$$= Ad_{a^{-1}} V$$
$$= Ad_{a^{-1}}\, \omega|_u (V^{\#}|_u).$$

Thus, two operators, $(R_a^*\omega)|_u$ and $Ad_{a^{-1}}\omega|_u$, have the same action on arbitrary vertical vectors. Let's see what they do to horizontal vectors:

$$(R_a^*\omega)|_u\,(H_u P) = \omega|_{ua}\,(R_{a*}(H_u P))$$
$$= \omega|_{ua}\,(H_{ua} P) = 0$$

$$Ad_{a^{-1}}\,\omega|_u\,(H_u P) = 0.$$

So they have the same action on all vectors, the operators are equal, and we have $R_a^*\omega = Ad_{a^{-1}}\omega$. To summarize the properties of the

Ehresmann form, we have:

$$\boxed{\begin{array}{ll} (1) & \omega|_u(H_uP) = 0 \\ (2) & \omega|_u(V^\#|_u) = V \\ (3) & R_a^*\omega = \mathrm{Ad}_{a^{-1}}\omega \end{array}}$$

Digression on the identity above. Let $g \in G$, $V \in \mathfrak{g}$. In the case of matrix groups, g and V are matrices, and $\mathrm{Ad}_g V = gVg^{-1}$. Then the identity,

$$g e^{tV} g^{-1} = e^{t(gVg^{-1})}$$

follows immediately from power series. (For matrix groups, $\exp(tV)$ is a genuine exponential.) For arbitrary Lie groups, $\mathrm{Ad}_g = I_{g*}|_e$, where $I_g : G \to G : a \mapsto gag^{-1}$ is the inner automorphism action of G on itself. Thus, $\mathrm{Ad}_g = L_{g*}R_{g^{-1}*}|_e$, since $I_g = L_g R_{g^{-1}}$. $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is the action of G on its own Lie algebra, the <u>adjoint representation</u>. Now note that $g\exp(tV)g^{-1}$ is a one-parameter subgroup,

$$(g\exp(sV)g^{-1})(g\exp(tV)g^{-1}) = g\exp(sV)\exp(tV)g^{-1}$$

$$= g\exp((s+t)V)g^{-1}.$$

So it must have the form $\exp tW$, for some $W \in \mathfrak{g}$. To find out what W is, compute the tangent vector at the identity,

$$W = [\exp(tW)] = [g\exp(tV)g^{-1}] = L_{g*}R_{g^{-1}*}[\exp tV]$$

$$= \mathrm{Ad}_g V.$$

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We see that a connection implies the existence of an Ehresmann form with properties (1)-(3) above. Note that $H_u P = \ker(\omega|_u)$. Conversely, suppose we are given a $\mathfrak{g}$-valued 1-form with properties (2) and (3). We then define $H_u P = \ker(\omega|_u)$. It then follows that $T_u P = V_u P \oplus H_u P$ and $R_{a*}(H_u P) = H_{ua} P$. (Exercise for you). Thus we have three equivalent ways of specifying a connection:

(1) Connection coefficients, $\Gamma^{\mu}_{\alpha\beta}$, or ∇ operator on M

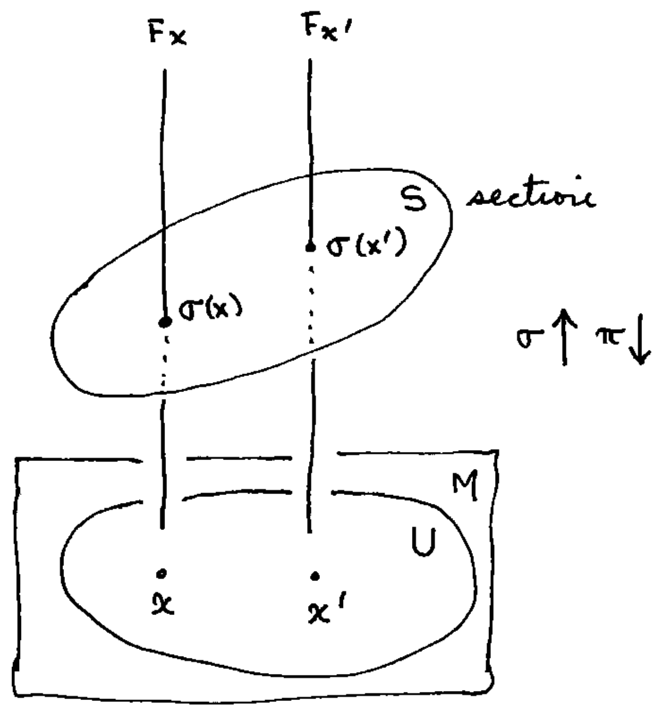
(2) G-invariant horizontal subspaces in PFB

(3) Ehresmann form ω.

Now we discuss a fourth equivalent way, which uses <u>gauge potentials</u>. A gauge potential A is a $\mathfrak{g}$-valued 1-form defined <u>on M</u>, <u>not P</u> (as ω). Actually, it is only defined over a local chart $U \subset M$, and the definition is made relative to a <u>local section</u> (in P) over U. It is thus intrinsically a coordinate-dependent object (i.e., section-dependent, but putting a local section in P is tantamount to choosing coordinates on P). For this reason we have to study how A transforms when we change local section (this is called a <u>gauge transformation</u>). Also, introducing a section complicates the geometry, as we will see. For all these reasons, purists (= mathematicians) prefer to work with ω, which has an intrinsic (geometrical) meaning on P, independent of coordinates. Physicists (as they say) prefer to work with gauge potentials, because often M is regarded as the "real" space we live on (i.e. space-time), where the potentials A are defined. (But one can question whether the "real" space is actually a PFB over M, where ω lives.) Also, vector potentials have a long tradition in physics.

A local section of a PFB is a map $\sigma : U \to P$, such that $\pi\sigma(x) = x$. A picture:

The surface S is the image of the map σ (we call also S the "section"). U and S are diffeomorphic, in fact the projection $\pi : P \to M$, restricted to S, is the inverse of σ. The section S need only be a smooth surface in P above U, which intersects each fiber transversally at one point. Thus there is a huge infinity of possible ways to choose a section, and anything dependent on it (like A or the coordinate g to be introduced momentarily) has a large degree of arbitrariness in it.

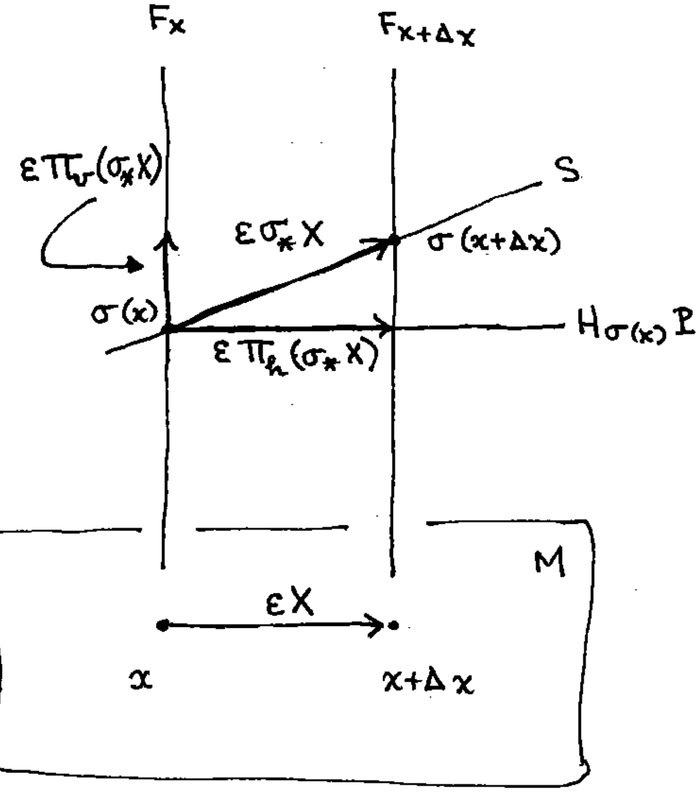


Supposing now that an Ehresmann form is given on P, we define

$$\boxed{A = \sigma^* \omega}$$

which makes A a field of maps, $A|_x : T_x M \to \mathfrak{g}$, for $x \in U$, i.e., $A \in \mathfrak{g} \otimes \Omega^1(U)$.

To interpret A, make a more schematic drawing of two nearby fibers in P and the section S. Look at fibers over x and $x+\Delta x$ in M. Write $\Delta x = \varepsilon X$, where $X \in T_x M$. $\sigma(x)$ is the point on the section over x, and the horizontal subspace $H_{\sigma(x)}P$ is drawn in. It is drawn at right angles on the paper, but this has no significance (in general there is no metric on P, even if there is one on M). The section is drawn at an angle to the horizontal subspace, because S is highly arbitrary and there is no reason why it (i.e., its tangent plane) should be horizontal.



By the definition of A and the pull-back, we have

$$A|_x(X) = (\sigma^* \omega)|_x(X) = \omega|_{\sigma(x)}(\sigma_* X).$$

But $\varepsilon \sigma_* X$ is the vector in $T_{\sigma(x)}P$ that is tangent to the section S and connects the same fibers as εX. Thus $\omega|_{\sigma(x)}$ acting on this projects out the vertical part and converts it to an element of $\mathfrak{g}$. In the diagram,

$$\Pi_v(\sigma_* X) = A|_x(X)^{\#}.$$

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Thus we can say that $A|_x(X)$ is a measure of the "slope" of the section S in the bundle at the point over x in the direction X, that is, $A|_x$ measures how much the section deviates from the horizontal at x. In particular, $A|_x = 0$ iff the section (i.e., its tangent plane) is purely horizontal at x.

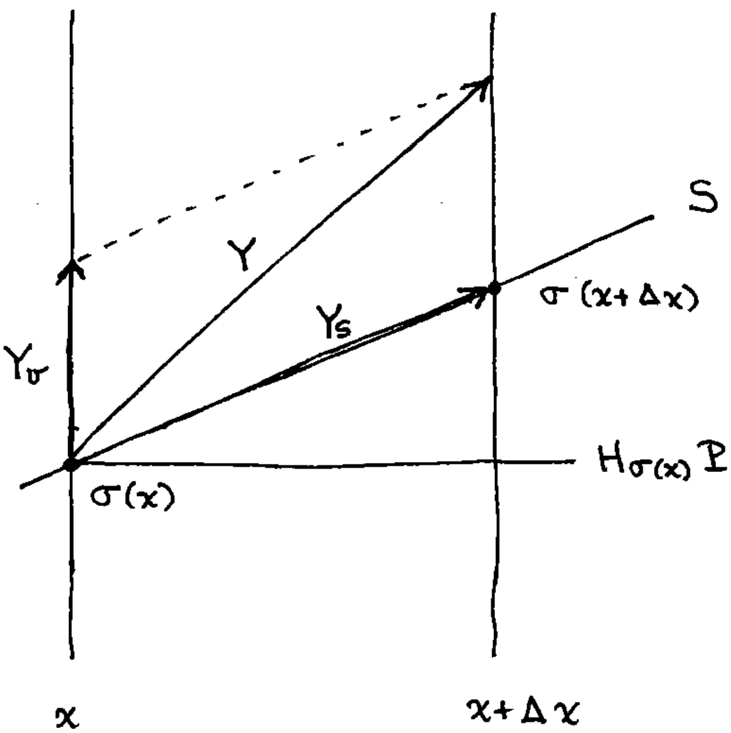
We can always choose a section S to be purely horizontal ~~at~~ over one point $x \in M$, but we cannot choose it to be horizontal over a region in M unless the distribution specified by H_uP is integrable. In that case, a purely horizontal section can be chosen, and $A = 0$ over the region in question.

The definition of A gives A in terms of ω. Can we find an expression giving ω in terms of A? The answer is yes, because by the property $R_a^* \omega = Ad_{a^{-1}} \omega$, to specify ω ~~it suffices~~ on a fiber it suffices to know ω (that is, to know its action on arbitrary tangent vectors) at any one point of the fiber. If A is given, the simplest point to choose (on F_x) is $\sigma(x)$.

In fact, by the formula above, the action of $\omega|_{\sigma(x)}$ on any vector tangent to S can be computed in terms of $A|_x$. Also, by the definition of ω, its action on vertical vectors is known. So, we need to take an arbitrary vector $Y \in T_{\sigma(x)}P$, break it into vertical and section (= tangent to section) components, and then $\omega|_{\sigma(x)}(Y)$ is computable. This is a different decomposition than that considered earlier (the horizontal-vertical) decomposition. In particular, when we write

$$Y = Y_s + Y_v$$

for the section and vertical components of Y, the vertical part Y_v is not the same as the Y_v we had earlier, when we wrote $Y = Y_h + Y_v$. This is clear from a picture: (You can see that Y_v is not the same as we would have in an h-v decomposition.)



So, we can write

$$\omega|_{\sigma(x)}(Y) = \omega|_{\sigma(x)}(Y_S) + \omega|_{\sigma(x)}(Y_v).$$

Of these, the first (s) term is easy, since if we write $X = \pi_* Y$, then X also $= \pi_* Y_S$, but $Y_S = \sigma_* X$, so

$$Y_S = \sigma_* \pi_* Y,$$

and

$$\omega|_{\sigma(x)}(Y_S) = \omega|_{\sigma(x)}(\sigma_* \pi_* Y)$$

$$= (\sigma^* \omega)|_x (\pi_* Y)$$

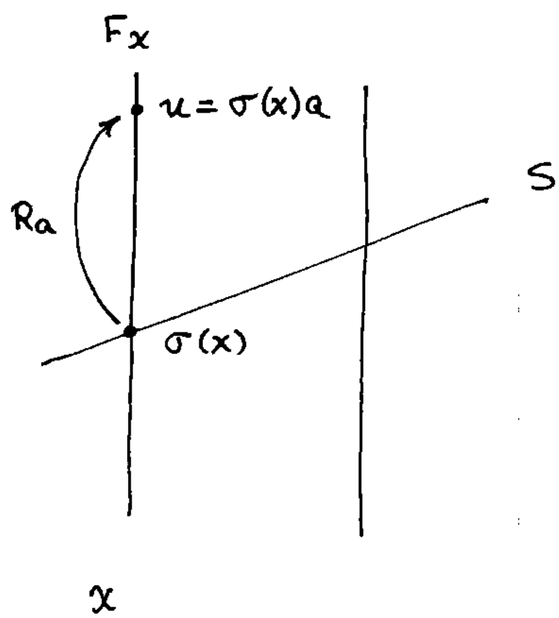
$$= A|_x (\pi_* Y).$$

For this term (the s-term), we just project Y onto M and use A on it.

For the other (v) term it helps to introduce explicit coordinates on the bundle P. If $u \in P$ (over $U \subset M$) we will write the coordinates of u as (x,a), where $x \in M$ and $a \in G$, by writing

$$u = \sigma(x)a, \qquad x = \pi(u)$$

Thus a is the group element needed to reach u from the section by the right action of G on P:



Obviously the a-coordinate of u depends on the section, while the x-coordinate does not. Let us write

$$g : \pi^{-1}(U) \to G$$
$$: u \mapsto a$$

for the coordinate function (distinguishing the function g from its value a). This is just another notation for the local trivialization associated with the section S,

$$\Phi : U \times G \to \pi^{-1}(U)$$
$$: (x,a) \longmapsto \sigma(x)a$$

or
$$\phi_{,x} : G \longrightarrow F_x$$
$$: a \longmapsto \sigma(x)a$$

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what Nak. calls the "canonical local triv.", so that

$$g(u) = \phi_{i,x}^{-1}(u) \quad , \quad \text{where} \quad x = \pi(u),$$
$$\text{or} \quad g(\sigma(x)a) = a.$$

Notice that the section S is given by $g(u) = e = \text{const.}$, i.e., $g(\sigma(x)) = g(\sigma(x)e) = e$.

Consider now the tangent map g_*. If we evaluate this at $u = \sigma(x)a$, we have

$$g_*|_u : T_u P \longrightarrow T_a G.$$

In particular, setting $u = \sigma(x)$, $a = e$, we get

$$g_*|_{\sigma(x)} : T_{\sigma(x)} P \rightarrow \mathfrak{g}.$$

Evaluated at $\sigma(x)$, g_* is a map very much like $\omega|_{\sigma(x)}$, i.e., it is a Lie-algebra-valued form. However, the kernel of g_* is the tangent space to the section $T_{\sigma(x)} S$ (the space of S-vectors), not $H_{\sigma(x)} P$ as is the case with $\omega|_{\sigma(x)}$. This is because g is constant on S, so $g_*(Y_S) = 0$ for any Y_S tangent to S.

As for vertical vectors, we have

$$g_*|_{\sigma(x)} (V^\#|_{\sigma(x)}) = V,$$

something which is almost obvious with a little intuition (think: how much does g vary when we move a small distance in a vertical direction away from $\sigma(x)$). But a formal proof is easily given using equivalence classes of curves for vectors:

$$g_*|_{\sigma(x)}\left(V^\#|_{\sigma(x)}\right) = g_*|_{\sigma(x)}\left[\sigma(x)\exp(tV)\right]$$

$$= \left[g\left(\sigma(x)\exp(tV)\right)\right]$$

$$= \left[\exp(tV)\right] = V.$$

$g_*|_{\sigma(x)}$ has the same action on vertical vectors as $\omega|_{\sigma(x)}$,

$$\omega|_{\sigma(x)}(Y_v) = g_*|_{\sigma(x)}(Y_v).$$

But, since $g_*|_{\sigma(x)}(Y_s) = 0$, the above is also $g_*|_{\sigma(x)}(Y)$.

Thus in our s-v decomposition, we have

$$Y = Y_s + Y_v \in T_{\sigma(x)}P$$

$$\omega|_{\sigma(x)}(Y) = \omega|_{\sigma(x)}(Y_s) + \omega|_{\sigma(x)}(Y_v)$$

$$= A|_x(\pi_* Y) + g_*|_{\sigma(x)}(Y)$$

$$= (\pi^* A)|_{\sigma(x)}(Y) + g_*|_{\sigma(x)}(Y).$$

This specifies the action of $\omega|_{\sigma(x)}$ on an arbitrary vector $Y \in T_{\sigma(x)}P$.

Now consider the action of ω at an arbitrary point $u = \sigma(x)a$. Let $Z \in T_uP$ be an arbitrary tangent vector at u. By the properties of ω we have

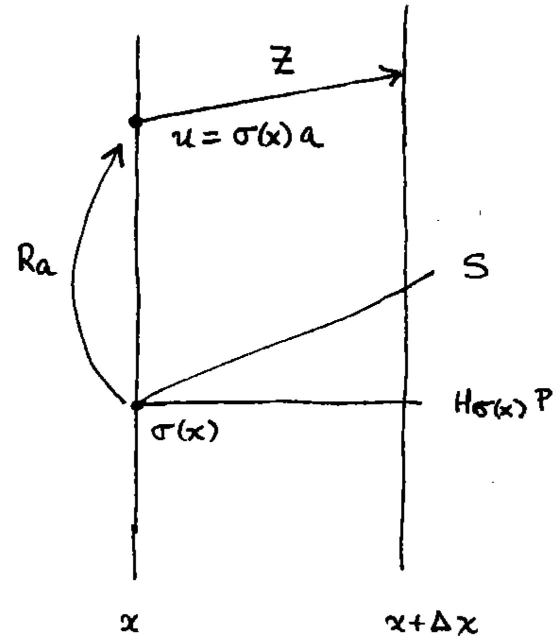
$$\omega|_u(Z) = \omega|_{\sigma(x)a}(Z) = (R_a^*\omega)|_{\sigma(x)}(R_{a^{-1}*}Z)$$

$$= Ad_{a^{-1}}\,\omega|_{\sigma(x)}(R_{a^{-1}*}Z).$$

So, we use the formula above for $\omega|_{\sigma(x)}$ and set

$$Y = R_{a^{-1}*} Z.$$

This gives,



$$\omega|_u(Z) = Ad_{a^{-1}}\left[A|_x (\pi_* R_{a^{-1}*} Z) + g_* R_{a^{-1}*} Z\right].$$

This can be simplified. First note that

$$\pi R_{a^{-1}} = \pi,$$

so

$$\pi_* R_{a^{-1}*} = \pi_*$$

and the $R_{a^{-1}*}$ can be dropped in the first term. Next, write an arbitrary element of P as $\sigma(x)b$ for some $b \in G$, and consider

$$\begin{aligned} g R_{a^{-1}}(\sigma(x)b) &= g(\sigma(x)ba^{-1}) \\ &= ba^{-1} \\ &= g(\sigma(x)b)a^{-1} \\ &= R_{a^{-1}} g(\sigma(x)b), \end{aligned}$$

i.e., g and $R_{a^{-1}}$ commute, so

$$g_* R_{a^{-1}*} = R_{a^{-1}*} g_*.$$

Finally, note that $Ad_{a^{-1}} = L_{a^{-1}*} R_{a*}$. Altogether, this gives

$$\omega|_u(Z) = Ad_{a^{-1}}\Big(A|_x(\pi^* Z)\Big) + L_{a^{-1}*}\, g_*\, Z,$$

or $$\boxed{\omega|_u = Ad_{a^{-1}}(\pi^* A)|_u + L_{a^{-1}*}\, g_*|_u,}$$

where it is understood that $a = g(u)$, $u = \sigma(x)a$. This is the explicit expression for ω in terms of A.

Nak. writes it this way:

$$\omega = g^{-1}(\pi^* A) g + g^{-1} dg.$$

He is not distinguishing between the function g and the value a, and he is thinking of a matrix group where $Ad_{a^{-1}} b = a^{-1} b a$ and where $L_{a^{-1}*}$ is just mult. by $a^{-1} = g^{-1}$. Also, we have never defined dg for a map like $g: \pi^{-1}(U) \to G$, but if you think of g as a "G-valued 0-form" then dg means the same as g_* (or, ~~thing~~ thinking of g as a matrix-valued function, then dg is a matrix of 1-forms).

Now it is possible to show that if A is a given $\mathfrak{g}$-valued 1-form on U, and if the section σ and associated function g are given, and if we define ω, a $\mathfrak{g}$-valued 1-form on $\pi^{-1}(U)$ by the boxed formula above, then ω satisfies the two requirements of an Ehresmann form,

$$\omega|_u(V^\#|_u) = V,$$

$$R_b^* \omega = Ad_{b^{-1}}\omega.$$

(Exercise for you). End of semester.