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Start today by computing the index (homotopy class of $\pi_1(S^1)$ of the transition function) of a specific $U(1)$ bundle over S^2, namely, the Hopf ~~map~~ fibration. The summary is the following:

$$E = P = S^3 \qquad (\text{principal fiber bundle})$$
$$M = S^2$$
$$F = G = U(1)$$

~~Right action of G on P~~ Let $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ be a 2-component spinor, $z \in \mathbb{C}^2 = \mathbb{R}^4$, or if normalized, $|z|^2 = 1$, then $z \in S^3 \subset \mathbb{R}^4$, as we henceforth assume. The (right) action of G on P ($U(1)$ on S^3) is

$$z \mapsto e^{i\alpha} z, \qquad z \in S^3, \quad e^{i\alpha} \in U(1).$$

It's just changing the "overall phase" of the spinor. It's conventionally taken to be a right action, but since $U(1)$ is Abelian you can apply it in any order. The projection $\pi : S^3 \to S^3/U(1) = S^2$ is defined by

$$\pi(z) = \langle z | \vec{\sigma} | z \rangle = \text{"the direction the spinor}$$

is pointing in", to use standard QM language. Call this $\hat{n}(z) \in S^2$. It's a unit vector. Use coordinates (θ, φ) on S^2.

To compute the homotopy class we need transition functions, and to compute those we need local trivializations ϕ_i or $\phi_{i,x}$. Notation:

$$\phi = \text{local trivialization}$$
$$\varphi = \text{azim. angle on sphere.}$$

since there are two phis.

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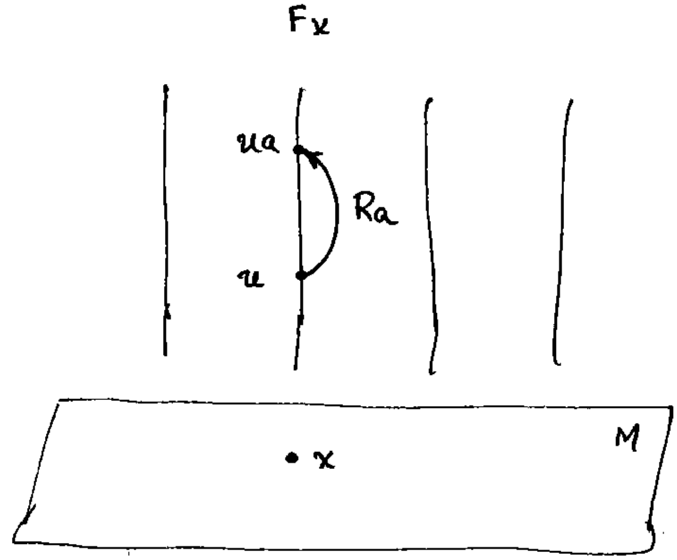
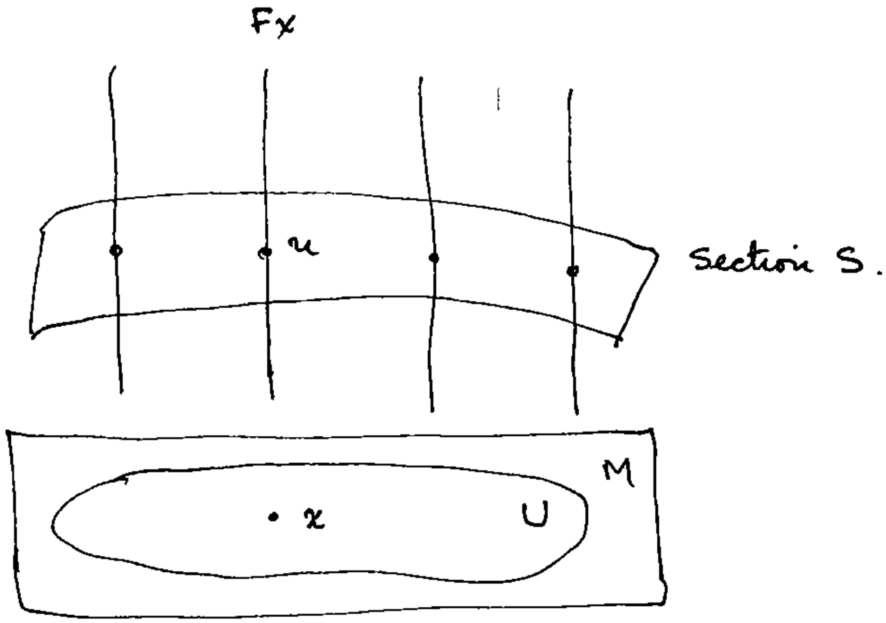
About local triv's in a P.F.B. (in general). The fibers are diffeo to the group G, but unlike a group they do not have an identity element. But if we take a fiber (over x, say) and arbitrarily ~~use~~ pick one point $u \in F_x$ as a reference point, then all other points of that fiber are accessible (and parameterizable) by means of the group element needed to reach them by the right ~~mult~~ action of G on P.



We denote the right action by $a \mapsto R_a$, where $a \in G$ and $R_a : P \to P$. It satisfies $R_a R_b = R_{ba}$, and $\pi R_a u = \pi u$ (the action is fiber preserving). We denote $R_a u$ by ua for short, where $u \in P$. In the picture above, point u on fiber F_x is identified with $e \in G$ and ua with $a \in G$. This gives us a map on one fiber,

$$\phi_{i,x} : G \to F_x : a \mapsto ua.$$

If we choose reference points like u in the picture above in a smooth manner over some region $U \in M$, the we get a local <u>section</u> of P, a surface of reference points,



Call the section $S: U \to P$, $\pi S(x) = x$. Then we get an associated local trivialization,

$$\phi : U \times G \to P : (x, a) \longmapsto S(x)\, a.$$

So on a P.F.B., a local section and a local trivialization imply one another.

For the Hopf map, the choice of a reference point on a fiber F_x is the same as a phase convention for the normalized spinors pointing in the direction $x = (\theta, \varphi)$. And the section S is a smooth assignment of phase conventions for spinors pointing in directions (θ, φ) over some region of S^2.

Let $z_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ be the spinor "pointing in" the $\hat{z}$-direction, i.e., $\langle z_0 | \vec{\sigma} | z_0 \rangle = \hat{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Then a spinor pointing in direction (θ, φ) can be written down using spin rotation operators,

$$z(\theta, \varphi) = R_z(\varphi)\, R_y(\theta)\, z_0,$$

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where the rotation about axis $\hat{a}$ by angle θ is

$$R_{\hat{a}}(\theta) = e^{-i\frac{\theta}{2}\hat{a}\cdot\vec{\sigma}} = \cos\frac{\theta}{2} - i(\hat{a}\cdot\vec{\sigma})\sin\frac{\theta}{2}.$$

So,

$$z(\theta,\varphi) = \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{+i\varphi/2} \end{pmatrix} \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\varphi/2}\cos\theta/2 \\ e^{i\varphi/2}\sin\theta/2 \end{pmatrix}.$$

This spinor ~~Eto~~ $z(\theta,\varphi)$ points in the (θ,φ) direction, but it is not continuous over the whole sphere, mainly due to the ½ angle $\varphi/2$ which gives a discontinuity on the line $\varphi=0$. We can get rid of this by multiplying by $e^{i\varphi/2}$ (a change in phase convention) to obtain the spinor

$$z_N(\theta,\varphi) = \begin{pmatrix} \cos\theta/2 \\ e^{i\varphi}\sin\theta/2 \end{pmatrix}.$$

This spinor is smooth everywhere on S^2 except at the south pole, where $e^{i\varphi}$ represents oscillations of finite amplitude that take place over a smaller and smaller region as $\theta \to \pi$. Thus the spinor is not differentiable at the south pole. The similar problem does not occur at the north pole, because $\sin\theta/2 \to 0$ there. So we call this spinor z_N to indicate that it is smooth over the northern hemisphere.

We get another phase convention if we multiply our original spinor by $e^{-i\varphi/2}$, giving

$$z_S(\theta,\varphi) = \begin{pmatrix} e^{-i\varphi}\cos\theta/2 \\ \sin\theta/2 \end{pmatrix},$$

which is smooth over the southern hemisphere but has a singularity at the north pole.

Our inability to find a phase convention that is smooth everywhere over S^2 is a sign that the Hopf bundle is nontrivial. But we can cover S^2 with two open sets U_N and U_S, as in the monopole vector potential, and we have the open cover necessary to make the Hopf bundle into a bundle according to the official definition. The spinors z_N and z_S defined above are local sections of the bundle, which imply local trivializations ϕ_N and ϕ_S. Explicitly,

$$\phi_N : U_N \times G \to \pi^{-1}(U_N)$$
$$: ((\theta,\varphi), e^{i\alpha}) \mapsto e^{i\alpha} z_N(\theta,\varphi) = e^{i\alpha}\begin{pmatrix} \cos\theta/2 \\ e^{i\varphi}\sin\theta/2 \end{pmatrix}$$

$$\phi_S : U_S \times G \to \pi^{-1}(U_N)$$
$$: ((\theta,\varphi), e^{i\alpha}) \mapsto e^{i\alpha} z_S(\theta,\varphi) = e^{i\alpha}\begin{pmatrix} e^{-i\varphi}\cos\theta/2 \\ \sin\theta/2 \end{pmatrix}.$$

There is only one transition function,

$$t_{NS,x} = \phi_{N,x}^{-1}\,\phi_{S,x}$$

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or

$$\phi_{S,x} = \phi_{N,x}\, t_{NS,x}$$

where $x \in U_N \cap U_S$, i.e., the equator where $\theta = \pi/2$, so write $x = (\frac{\pi}{2}, \varphi)$ or just φ. Then putting $\theta = \pi/2$, we get

$$\frac{e^{i\alpha}}{\sqrt{2}} \begin{pmatrix} e^{-i\varphi} \\ 1 \end{pmatrix} = \frac{e^{i\alpha}}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\varphi} \end{pmatrix} t_{NS}(\varphi),$$

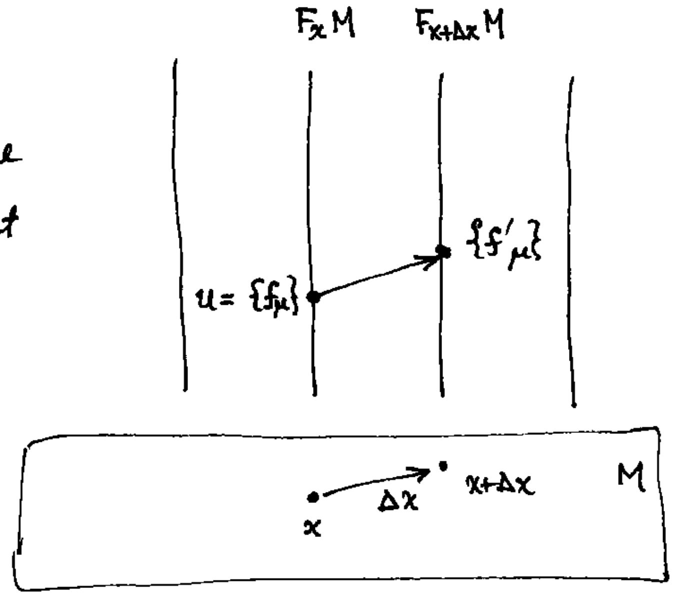
or

$$t_{NS}(\varphi) = e^{-i\varphi}.$$

The transition functions belong to the homotopy class $n=-1$ in $\pi_1(S^1)$. Thus the Hopf bundle is not trivial, as we suspected.

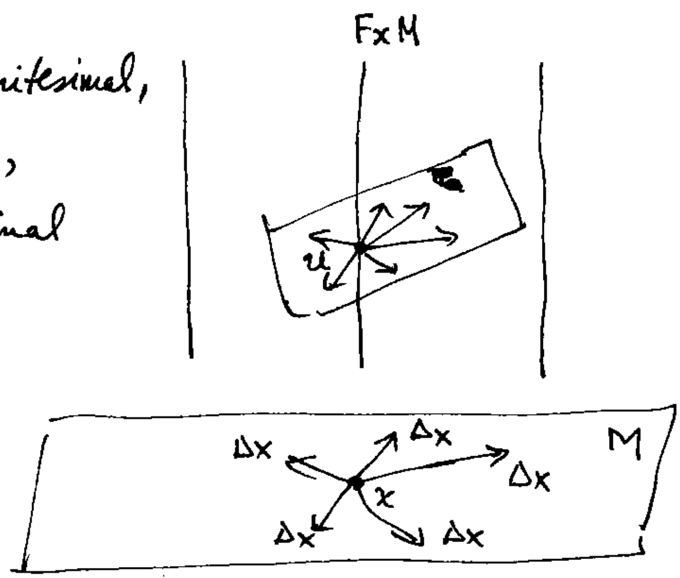
Now we turn to the subject of <u>connections</u>. The example we know of connections concerned the parallel transports of tangent vectors, i.e., in TM. The problem of parallel transport occurs in other vector bundles, too, for example the H.L.B. on which ψ (quantum state) lives. Without a definition of parallel transport (a connection) it is impossible to define a derivative (covariant deriv) operation, and thus impossible to have differential equations (field equations) that have a geometrical meaning. For example, a Schrödinger equation for ψ. The natural setting for understanding connections is a P.F.B. It turns out connections on a P.F.B. are equivalent to gauge <u>potentials</u> (vector potentials) on the base space.

Begin with the example we know, connections on TM. We introduced these originally for defining infinitesimal parallel transport of tangent vectors, i.e., creating an identification of nearby tangent spaces T_xM and $T_{x+\Delta x}M$. Now let's view this in the frame bundle FM.



Let $\{f_\mu\}$ be a specific frame at $x \in M$, i.e., a point in the frame space F_xM over x. This is not a field, just one frame. By parallel transporting each of the vectors that makes up the frame (f_μ) over to a nearby tangent space at $x+\Delta x$, we get a new frame, call it $\{f'_\mu\}$. This creates a small displacement vector <u>in the bundle</u> (betw. $\{f_\mu\}$ and $\{f'_\mu\}$) corresponding to a given small displacement vector Δx <u>in the base space</u>. If we let Δx range over all nearby tangent spaces, the corresponding vector in the bundle sweeps out a surface.

Since the vectors are infinitesimal, this surface lies in T_uP, where $u = \{f_\mu\}$ = the original given frame and $P = FM$ (a P.F.B.).



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Thus we get a map: $T_x M \to T_u P$, where $u \in F_x M$, defined for any $u \in F_x M$, once we have a connection. This map is linear, because ∥ transport is linear in Δx, so it specifies a vector subspace of $T_u P$. Notice the dimensions,

$$\dim T_x M = \dim M$$
$$\dim T_u P = \dim P = \dim M + \dim G.$$

The structure group $G = GL(m, \mathbb{R})$ or maybe $SO(m)$ or some other group. So the map maps a smaller vector space to a larger one. The kernel of the map is $\{0\}$, since if Δx is nonzero, then it connects two distinct points of M, and the corresponding vector in $T_u P$ must be nonzero, because it joins two distinct fibers (over x and $x+\Delta x$). Therefore $\dim \operatorname{im}(\overset{\text{our unnamed}}{\text{map}}) = \dim M$, the map has maximal rank. The subspace of $T_u P$ swept out by the process described is isomorphic to $T_x M$.

The subspace described ($\operatorname{im} T_x M$ under our map) is called the <u>horizontal subspace</u> at $u \in P$, denoted $H_u P$.

One can also define a <u>vertical subspace</u> $V_u P$ at $u \in P$. It is just the tangent space ~~at~~ at u to the fiber,

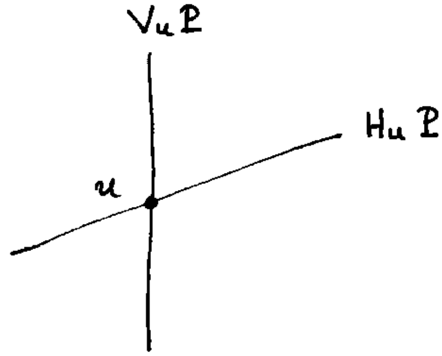
$$V_u P = T_u (F_x), \quad \text{where} \quad x = \pi(u).$$

Since $\dim F_x = \dim G$, we have

$$\dim V_u P = \dim G.$$

Any bundle has a vertical subspace, but we get a horizontal subspace on a P.F.B. only if there is a connection.

We sketch it like this,



at some angle because we don't wish to imply that $H_u P$ is orthogonal to $V_u P$ (in general there is no metric on P, even if there is one on the base space M). However, $H_u P$ <u>is</u> transverse to $V_u P$, meaning

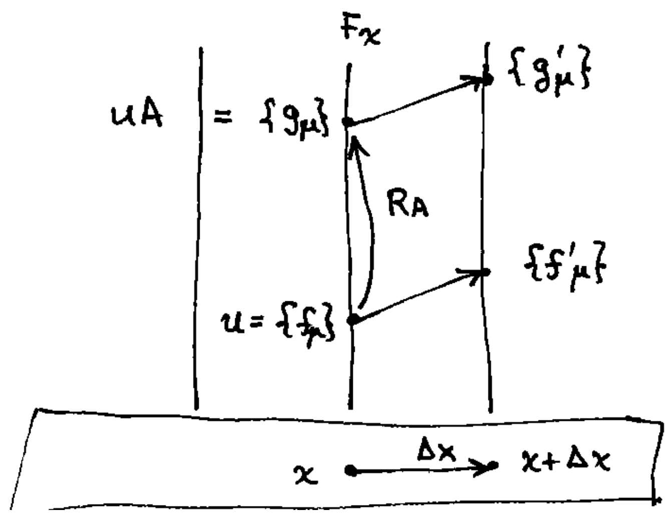
$$T_u P = H_u P \oplus V_u P$$

i.e., any vector in $T_u P$ can be represented uniquely as the sum of a horizontal and a vertical vector.

The horizontal subspaces have another property which we see if we parallel transport another frame $\{g_\mu\} \in F_x M$, where

$$g_\mu = f_\nu A^\nu{}_{\cdot\mu}, \qquad A \in G \qquad (A = \text{some matrix}).$$

This is the right action of G on P in the case of a frame bundle; we can write R_A for this action. The picture:



What is the relation between $\{f'_\mu\}$ and $\{g'_\mu\}$? Because ∥ transport is linear in the vector transported, we have

$$g'_\mu = f'_\nu \; A^\nu{}_\mu$$

(the parallel transport does not affect the matrix A). That is,

$$\{g'_\mu\} = R_A \{f'_\mu\}$$

if

$$\{g_\mu\} = R_A \{f_\mu\}.$$

Parallel transport and right action commute. Thus, if ~~$X \in H_u P$, then $R_{a*} X \in H_{ua}$~~ $Y \in H_u P$, then $R_{a*} Y \in H_{ua} P$. In fact since R_a is a diffeomorphism (R_{a*} has full rank), we have

$$R_{a*}(H_u P) = H_{ua} P.$$

We now define a <u>connection</u> on a P.F.B. (P, M, G, π) as a smooth assignment of "horiz." subspaces $H_u P \subset T_u P$ at each $u \in P$ such that

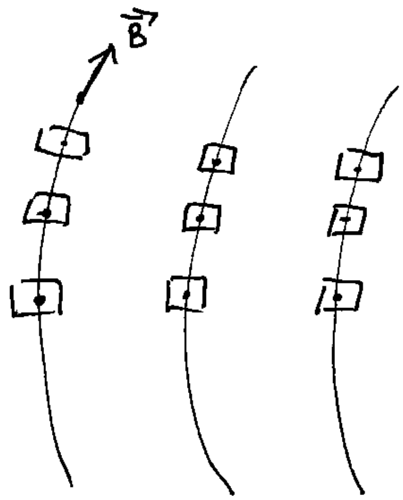
$$\text{(a)} \quad T_u P = H_u P \oplus V_u P$$

$$\text{(b)} \quad R_{a*}(H_u P) = H_{ua} P.$$

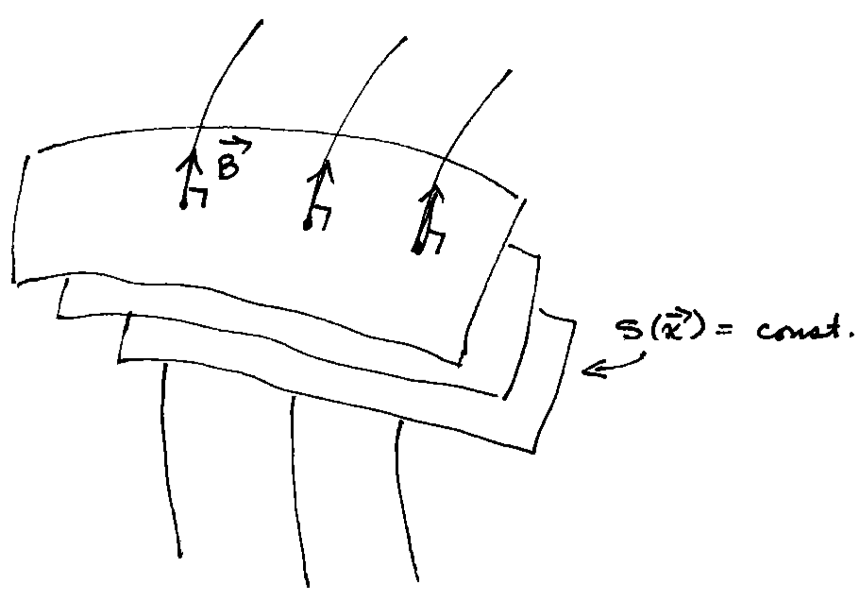
(General defn for any P.F.B. Later we give another, equivalent defn.)

Now we make a digression. On any manifold M, we define an <u>r-distribution</u> Δ as a smooth assignment of r-dimensional subspaces $(0 \le r \le \dim M)$ in each tangent space $T_x M$.

For example, consider $M \subseteq \mathbb{R}^3$, a region in which a nonzero magnetic field $\vec{B}$ is defined. The field has field lines (the integral curves of $\vec{B}$). At each point of this region we consider the planes (or small pieces of planes) $\perp$ to $\vec{B}$.



The planes really live in the tangent spaces. A natural question is whether these little pieces of planes can be glued together smoothly to form surfaces (actually, a family of surfaces). If so, $\vec{B}$ is everywhere $\perp$ to the surfaces.



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By assigning values to these surfaces (const on each surface), we can regard the surfaces as level sets of a scalar $s(\vec{x})$. Then since $\vec{B} \perp$ surfaces, we have

$$\vec{B} = \mu \nabla s$$

where μ is some scalar (it can depend on $\vec{x}$). But this implies

$$\nabla \times \vec{B} = \nabla \mu \times \nabla s,$$

$$\vec{B} \cdot (\nabla \times \vec{B}) = 0.$$

So such surfaces exist only if $\vec{B} \cdot \nabla \times \vec{B} = 0$. In fact, it is iff, $\vec{B} \cdot \nabla \times \vec{B} = 0$ is the integrability condition for the existence of scalars μ and s in $\vec{B} = \mu \nabla s$, given $\vec{B}$. We see that in general, 2D surfaces orthogonal to a vector field in 3D do not exist.

We say that an r-distribution Δ on M is <u>integrable</u> if there exists (locally) a foliation of M into r-dimensional submanifolds such that at each x (in some local region of M) Δ_x is tangent to the manifold passing through x.

We say that a vector field $X \in \mathfrak{X}(M)$ lies in a distribution Δ if $X|_x \in \Delta|_x$ at each x. If Δ is integrable, then ~~the~~ any integral curves of $X \in \Delta$ must lie in one of the submanifolds to which Δ is tangent (a fairly obvious geometrical fact). Thus, by following integral curves of vector fields lying in an integrable distribution Δ, we can explore the corresponding submanifolds. ~~These~~ Such vector fields need not commute, but if the distribution is integrable, then $[X, Y]$ should lie in Δ. (Another fairly intuitive fact.) These ideas make plausible the following theorem:

> Thm. A distribution Δ is integrable iff $X, Y \subset \Delta$ (X, Y vector fields on M) implies
>
> $$[X, Y] \subset \Delta.$$

This is the Frobenius theorem. It has several versions. If we specify an r-distribution by r linearly indep. vector fields $X_1, \dots, X_r$, then Δ is integrable iff

$$[X_i, X_j] = C^k_{ij} X_k,$$

where the C^k_{ij} are allowed to be functions of position. (Thus, the $\{X_i\}$ do not usually form a Lie algebra.)

To return to connections, we can say that a connection on a P.F.B. is an m-dimensional distribution on P ($m = \dim M$), invariant under the group action and transverse to the fibers.

In general, this distribution which defines a connection is not integrable. One can show that it is integrable iff the curvature tensor (on M) vanishes.