5/4/04

The tangent bundle TM and frame bundle FM have the same basespace M, and same structure group $GL(m,\mathbb{R})$. When we showed that these spaces actually are bundles, we used the same open cover $\{U_i\}$ of M for both these bundles, and we found that the transition functions $t_{ij}: U_i \cap U_j \to G$ were the same, that is, we found $t_{ij}(x) = J(x)$ where J is the Jacobian matrix connecting the i-coordinates with the j-coordinates on a fiber. The fibers, however, are different ($\mathbb{R}^m$ for TM; $GL(m,\mathbb{R})$ for FM).

A bundle was defined as trivial if it is possible to gauge away the transition functions, i.e., to find functions $g_i : U_i \to G$ for all i such that

$$g_i(x)^{-1}\, t_{ij,x}\, g_j(x) = e, \quad \forall\, x \in U_i \cap U_j \quad \forall\, i,j.$$

The possibility of doing this depends on the sets $\{U_i\}$ and functions t_{ij}, but not on the nature of the fiber. Thus, in the case of TM and FM, if we can gauge away t_{ij} for one bundle we can do it for the other, and TM is trivial iff FM is trivial. This is another point of view on the theorem just recently proved.

Bundles TM and FM are said to be <u>associated</u>, meaning they have the same M, G, $\{U_i\}$ and t_{ij}, but different fibers. Let us consider the problem of constructing a new bundle associated to a given (original) one, in which the fiber changes. Denote the properties of the original bundle with a 0-subscript, and the new bundle without a 0 subscript. Then we have:

5/4/04

original: $E_0 \quad M \quad F_0 \quad G \quad \pi_0 \quad \{U_i\} \quad \{\varphi_{i0}\} \quad t_{ij}$

new: $M \quad F \quad G \quad \{U_i\} \quad t_{ij}$

We drop the 0-subscript on objects common to both bundles, M, G, U_i, t_{ij}, but note the fiber has changed from F_0 to F. This gives us partial information about the new bundle. Can we fill in the missing elements $(E, \pi, \{\varphi_i\})$?

Alternatively, we might image that someone has given us partial information about a bundle (the info on the 2nd line above), and asks us to reconstruct the bundle. This is the <u>reconstruction</u> problem.

We begin by constructing the locally trivial Cartesian products $U_i \times F$, and considering the disjoint union of these:

$$X = \bigcup_{\substack{i \\ \text{disjoint}}} U_i \times F .$$

This means the following. An element of X is a triplet,

$$(i, x, f), \qquad \text{where} \quad x \in U_i, \ f \in F.$$

That is, points of X remember which U_i they came from, and

$$(i,x,f) = (j, x', f') \quad \text{iff} \quad \begin{matrix} i = j \\ x = x' \\ f = f' \end{matrix}$$

Now define a relation on X,

$$(i, x, f) \sim (j, x', f')$$

if $\quad x = x'$
$\quad f = t_{ij,x} f'$

This is an equivalence relation if $t_{ij,x}$ satisfies:

(a) $t_{ii,x} = id_F \quad (\text{or } e \in G), \qquad x \in U_i$

(b) $t_{ij,x}^{-1} = t_{ji,x}, \qquad x \in U_i \cap U_j$

(c) $t_{ij,x}\, t_{jk,x} = t_{ik,x} \qquad x \in U_i \cap U_j \cap U_k.$

$t_{ij,x}$ does satisfy these conditions if it came from some original bundle. If not, these are extra conditions that the $t_{ij,x}$ have to satisfy in order to (re)construct the new bundle.

The equivalence relation above amounts to using glueing rules for the regions U_i that reproduce the glueing rules in the original bundle (because the t_{ij} are the same). Both bundles are given the same "twists".

Now define

$$E = \frac{X}{\sim},$$

so $\qquad$ element of $E = [(i, x, f)]$.

Then define

$$\pi : E \to M$$
$$: [(i, x, f)] = x.$$

This is meaningful, since all elements (i, x, f) of the equivalence class have the same x. The fiber over x_0 is

$$F_{x_0} = \pi^{-1}(x_0) = \{ [(i, x, f)] \mid x = x_0 \}.$$

It is diffeo. to F because $x = x_0$ is fixed, $f \in F$, and different i's are related by the equivalence relation.

Finally, define

$$\phi_i : U_i \times F \to \pi^{-1}(U_i)$$

$$\text{or } \phi_{i,x} : F \to F_x$$
$$: f \mapsto [(i,x,f)].$$

The obvious definition. Can check that $t_{ij,x} = \phi_{i,x}^{-1} \phi_{j,x}$ (the final step), and then we have reconstructed the bundle.

The original bundle could be a vector bundle (F_0 = some vector space), in which case we might choose $F = G$ to get the associated P.F.B. This will be isomorphic to the frame bundle. Or the original bundle might be a P.F.B. and new bundle a vector bundle, in which case we get vector bundles associated with the P.F.B. In this case, G usually acts on F (the vector space) by some representation. In this way we can construct the cotangent bundle and all the various tensor bundles as bundles associated with FM.

An interesting case is when one bundle is FM in GR with orthonormal frames, so that $G = L_0 \in O(1,3)$ (proper orthochronous Lorentz transformations), and we wish the associated bundle to be a spin bundle with $F = \mathbb{C}^2$ (Weyl spinors) or $\mathbb{C}^4$ (Dirac spinors). The subtlety in this case is that the structure group must be lifted to $SL(2,\mathbb{C})$ (it is not the same structure group), and as a result spin bundles do not exist over just any space-time manifold.

We now consider the behavior of bundles under maps. It turns out that bundles can be pulled back, but not generally pushed forward.

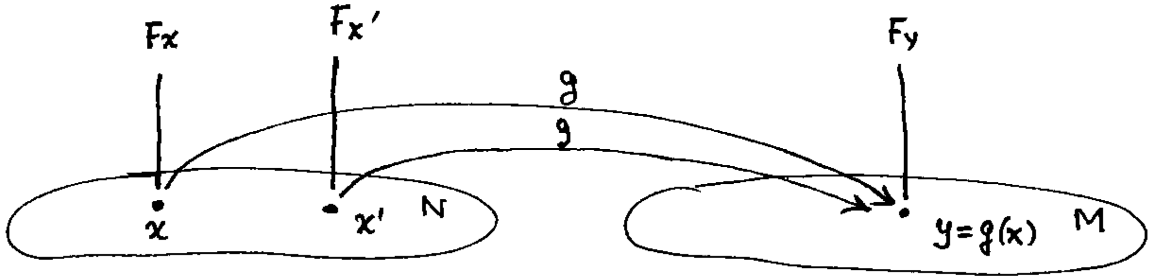
Let $E \xrightarrow{\pi} M$ represent a bundle over M with std. fiber F, and let $g: N \to M$ be a map. Here are the spaces:

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ N \arrow[r, "g"] & M \end{tikzcd}$$

It turns out that we can pull back the bundle structure over M to create a new one over N; (N will have the same std. fiber F as M).

$$\begin{tikzcd} \tilde{E} \arrow[d, "\tilde{\pi}"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "g"] & M \end{tikzcd}$$

where $\tilde{E}, \tilde{\pi}$ denote the new bundle $\tilde{E} \xrightarrow{\tilde{\pi}} N$. The idea is that fibers over M get pulled back and copied to make fibers ~~of~~ over N.



In the picture, F_x will be made an identical copy of F_y, where $y = g(x)$. Note that g does not have to be injective. There may be more than one point of N $(x, x'$ above) that map to a given $y \in M$. If so, both fibers $F_x, F_{x'}$ are identical copies of F_y. By pulling back M fibers to N at all points $x \in N$, we get a new bundle over N. This is the intuitive idea of the pull-back bundle.

By "identical copy" we mean that there is a natural diffeo~~iso~~morphism (based on the given geometrical elements) between F_x and F_y in the diagram above (and betw. $F_{x'}$ and F_y). Of course all fibers are diffeo. to the std. fiber F and hence to each other, but not usually in a natural way. This natural diffeomorphism between F_x for any $x \in N$ and F_y for $y = g(x)$ amounts to a fiber-preserving diffeo betw. $\tilde{E}$ and E, call it $\bar{g}: \tilde{E} \to E$ (the *lift* of $g: N \to M$), so the overall picture is

$$\begin{tikzcd} \tilde{E} \arrow[r, "\bar{g}"] \arrow[d, "\tilde{\pi}"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "g"] & M \end{tikzcd}$$

This is a commuting diagram (since $\bar{g}$ is fiber preserving).

Now to actually construct the pull-back bundle. Given data:

Orig. (over M):	E.	M	F	G	π	U_i	ϕ_i	t_{ij}
new (over N):		N	F	G				

Initially all we know is the base space N and std fiber F and structure group (the latter, two assumed to be the same for both bundles).

First let us get the open cover for N. Define $V_i = g^{-1}(U_i)$. Since g is continuous (as we assume), the inverse images of open sets are open, and the V_i are open sets on N. This is one reason why pushing forward a bundle won't work in general, the forward image of an open set is not necessarily open. Moreover, the collection $\{V_i\}$ forms an open cover of N, since every $x \in N$ lies in some V_i (since $f(x) \in M$ lies in some U_i).

Next get the transition functions for the new bundle. Call them s_{ij}. Then we have

$$t_{ij} : U_i \cap U_j \to G$$

$$s_{ij} : V_i \cap V_j \to G.$$

Noting that $g^{-1}(U_i \cap U_j) = V_i \cap V_j$, it is logical to define s_{ij} as the pullback of t_{ij},

$$s_{ij} = g^* t_{ij},$$

i.e. $$s_{ij,x} = s_{ij}(x) = t_{ij}(g(x)) = t_{ij,g(x)}.$$

We now have everything needed $(N, F, G, \{V_i\}, s_{ij})$ to proceed with the reconstruction program, giving us $\tilde{E}, \tilde{\pi}$, and ψ_i (the local trivializations).

Following the reconstruction program above, with changes of notation, we have

$$X = \bigcup_{\substack{i \\ \text{disjoint}}} V_i \times F$$

$(i, x, f) \in X$ where $x \in V_i$, $f \in F$

$(i, x, f) \sim (j, x', f')$ if $x = x'$, $f = s_{ij,x} f'$.

Note that the $s_{ij,x}$ satisfy the consistency requirements (a)(b)(c) since the $t_{ij,y}$ do. Then

$$\tilde{E} = \frac{X}{\sim},$$

$[(i, x, f)] \in \tilde{E}$,

$\tilde{\pi} : \tilde{E} \to N : [(i, x, f)] \mapsto x.$

$\psi_{i,x} : F \to F_x : f \mapsto [(i, x, f)].$

5/4/04

So we have reconstructed the pull-back bundle. Denote the original bundle (over M) by E for short, and the pull-back bundle by $g^* E$.

Finally, to define the lift $\bar{g}: \tilde{E} \to E$, define

$$\bar{g}: [(i,x,f)] = \phi_{i,g(x)} f.$$

This maps F_x diffeomorphically onto F_y, where $y = g(x)$. Can check that this def'n of $\bar{g}$ is independent of the representative element of the equivalence class.

Now consider the same basic picture, but with two maps $f, g: N \to M$:

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ N \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & M \end{tikzcd}$$

So we get 3 bundles, E over M and f^*E, g^*E over N. Then we have a theorem:

<u>Thm</u>: If f is homotopic to g, then f^*E is equivalent (i.e. compatible with) to g^*E.

No proof here. Intuitively, this says that if f^*E continuously changes into g^*E, then the topology of the bundle can't change either.

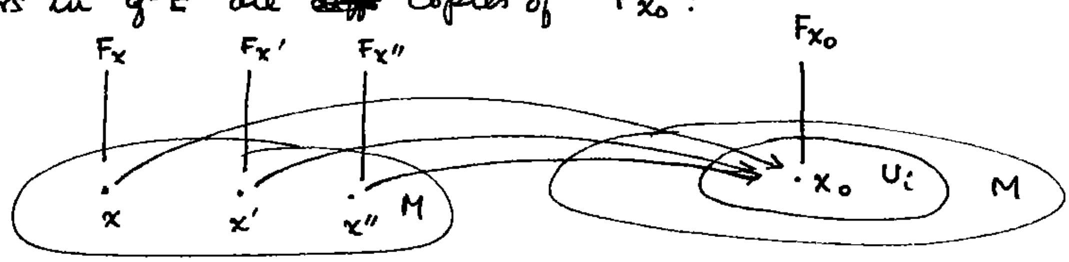
Another variation on the above. Set $N = M$, then

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ M \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & M \end{tikzcd}$$

gives us 3 bundles over M, E, f^*E, g^*E. Now let

$$f: M \to M: x \mapsto x, \qquad f = id_M$$
$$g: M \to M: x \mapsto x_0 \qquad \text{const. map.}$$

Since $f = id_M$, $f^*E = E$. As for g, since it is a constant map, all fibers in g^*E are ~~diff~~ copies of F_{x_0}:



So there is a natural isomorphism of each fiber ~~two~~ of g^*E with one fiber F_{x_0}, hence there exists a global trivialization of g^*E, hence g^*E is trivial. Another way to see the same thing is to note that if $x_0 \in U_i$, then $g^{-1}(U_i) = M$. Thus, $V_i = M$, and M is covered (for the purpose of bundle g^*E) by a single chart. Again, this implies a trivial bundle (the local triv. on this chart is a global triv.)

Now suppose M is <u>contractible</u>, so $f \sim g$. Then by the previous thm we have a new thm: (useful and important.)

Thm. If M is contractible, then any bundle over M is trivial.

Now consider how bundles get used in quantum mechanics and field theory. Let ψ be the wave function for a nonrelativistic charged spinless particle. We usually think of ψ as a complex-valued scalar field:

$$\psi: \mathbb{R}^3 \to \mathbb{C}.$$

Notice that $|\psi|^2$ is physically measurable at a point $\vec{x}$ of space, since it

5/4/04

is just the probability density there. However, there are several subtleties to the problem of measuring the phase of ψ.

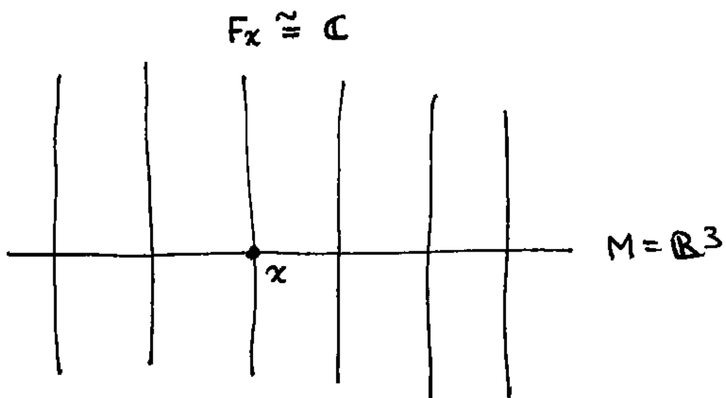
One is that the phase of ψ is not even determined until we specify a convention for the vector potential. This is because when we do a gauge transformation on $\vec{A}$, ψ changes also:

$$\vec{A}' = \vec{A} + \nabla\chi \qquad \chi = \text{"gauge scalar"}$$

$$\psi'(\vec{x}) = e^{-\frac{ie}{\hbar c}\chi(\vec{x})}\,\psi(\vec{x}).$$

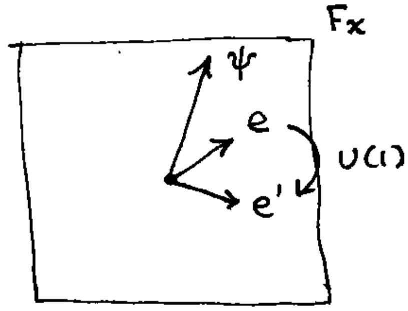
ψ changes by a phase factor that depends on $\vec{x}$, thus the phase of ψ cannot be definite without a definite convention for $\vec{A}$.

This situation is clarified if we introduce a bundle with $M = \mathbb{R}^3$ (physical space), and $F = \mathbb{C}$. This is an example of a <u>Hermitian line bundle</u>. A line bundle in general is any vector bundle in which the fibers are



1D vector spaces (real or complex). Here they are complex. The "Hermitian" part means that we introduce a complex metric onto each fiber, that allows us to compute $|\psi|^2$ at each point x. Since each fiber has a metric, we restrict consideration to orthonormal frames, so the structure group is $U(1) = G$. (We would use $U(n)$ if $F = \mathbb{C}^n$.)

We ignore the possibility that the individual fibers F_x might have a natural isomorphism with $F = \mathbb{C}$, since on a general bundle there would be none, and think of F_x as a plane (a 1D complex vector space) with an origin (a 0-vector) but no prefered basis. A basis would consist of one nonzero vector in F_x, call it e. We choose it to be a unit vector. Given the basis e, an arbitrary vector in F_x, call it ψ, can be represented as a linear combination of e with a complex coefficient, call it $\psi(x)$:



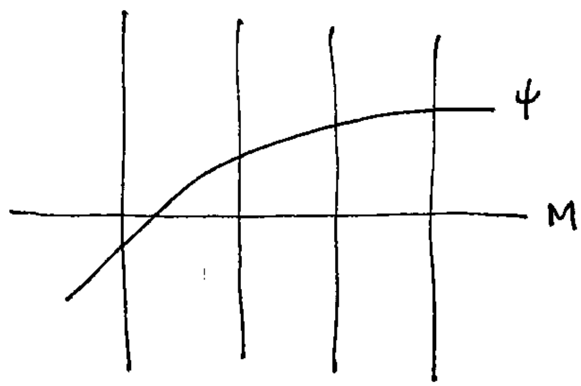
$$\underset{\substack{\uparrow \\ \text{abstract} \\ \text{vector}}}{\psi} = e \underset{\substack{\uparrow \\ \text{its component} \\ \text{w.r.t. } e}}{\psi(x)}$$

We can of course change basis, $e \to e'$ by a $U(1)$ transformation, whereupon the vector ψ acquires a new component $\psi'(x)$.

We interpret the equation for the transformation of ψ under a gauge transformation as giving two components, $\psi(x)$ and $\psi'(x) \in \mathbb{C}$, of one abstract vector ψ w.r.t two different bases e and e', related by the x-dependent $U(1)$ phase factor $e^{-\frac{ie}{\hbar c}\chi(\vec{x})}$. Then the quantum state of the particle is represented by a section of the Hermitian line bundle,

5/4/04

$F = \mathbb{C}$



This is a global section, since ψ should exist everywhere. This is a pleasing picture, because the section does not depend on the gauge.

We must suppose somehow that a field of frames in the frame bundle, called e or e' above, is specified once a gauge convention has been chosen for $\vec{A}$. The precise manner in which this occurs will be explained later. We just note for now that a section of the frame bundle is in general only locally defined, since a global section of the frame bundle exists iff the frame bundle is trivial. Thus, the wave function $\psi(x)$ will be only locally defined, in general, even though the section ψ is global.

In the case that $M = \mathbb{R}^3$ (or $\mathbb{R}^4$ for a time-dependent problem), all the bundles discussed above are trivial, since $\mathbb{R}^3$ or $\mathbb{R}^4$ is contractible. In the case of a trivial bundle, the global section ψ can be described by a global wavefunction $\psi(x)$, and we return to the usual way of thinking about wave functions, as complex-valued fields,

$$\psi : \mathbb{R}^3 \rightarrow \mathbb{C} .$$

A <u>field</u> in the usual sense is just a mapping,

$$\psi : M \rightarrow F$$

where usually $M \subseteq \mathbb{R}^3$ or $\mathbb{R}^4$ and F is the "field value space", for example, $\mathbb{R}^3$ for an electric field $\vec{E}$. Thus a field in this case is just a special case of a map between two spaces. Write such a map generally as $f: A \to B$. The map f can be seen geometrically as the set of points $(a, b) \in A \times B$ such that $b = f(a)$. This is just the graph of the function, but it can also be seen as a global section of the trivial fiber bundle $A \times B$. Thus, a global section of a possibly nontrivial fiber bundle is a generalization of the usual notion of a function or map between two spaces, or of a field in physical application.

An application where the Hermitian line bundle carrying the section ψ is nontrivial occurs when we have a charged particle moving in the field of a magnetic monopole,

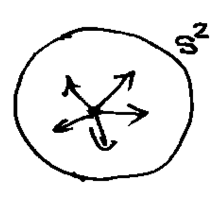
$$\vec{B} = g\frac{\hat{r}}{r^2}.$$

We remove the origin to avoid the singularity, and set $M = \mathbb{R}^3 - \{0\}$. This is not a contractible space. The magnetic field 2-form is

$$B = \tfrac{1}{2}\varepsilon_{ijk} B_i \, dx^j \wedge dx^k,$$

it satisfies $dB = 0$, which is equivalent to $\nabla\cdot\vec{B} = 0$. But $B \neq dA$, i.e., there does not exist a global, smooth vector potential for $\vec{B}$. This is because a closed form is exact iff its integral over all cycles vanishes. But letting S^2 be any sphere surrounding the monopole, we have

$$\int_{S^2} B = 4\pi g,$$

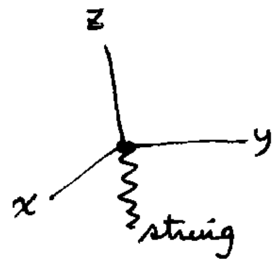


so B is not exact.

If we write $\vec{B} = \nabla \times \vec{A}$ in spherical coordinates, it is easy to uncurl $\vec{B}$ and find $\vec{A}$. The solution is not unique, but one that emerges is

$$\vec{A}_N = g \, \frac{(1-\cos\theta)}{r \sin\theta} \, \hat{\varphi} = g \, \frac{\tan\theta/2}{r} \, \hat{\varphi}$$

This $\vec{A}$ is singular on the negative z-axis, where $\tan\theta/2 \to \infty$ and $\hat{\varphi}$ is undefined. But it is smooth on the positive z-axis, since $\tan\theta/2 \to 0$ and the lack of direction of $\hat{\varphi}$ is not a problem. The (half) line of singularities of $\vec{A}$ (the neg. z-axis) is called the <u>string</u> of the monopole, although it should be called the string of the vector potential.

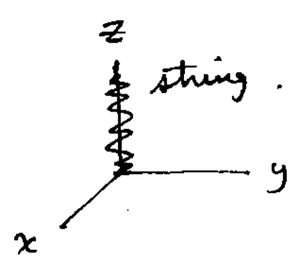


Since this $\vec{A}$ is smooth and nonsingular over the northern hemisphere, we write $\vec{A}_N$ for it.

We can get an $\vec{A}$ that is nonsingular over the southern hemisphere by uncurling $\vec{B}$ in a slightly different manner. This gives

$$\vec{A}_S = g \, \frac{(-1-\cos\theta)}{r \sin\theta} \, \hat{\varphi} = -g \, \frac{\cot\theta/2}{r} \, \hat{\varphi}.$$

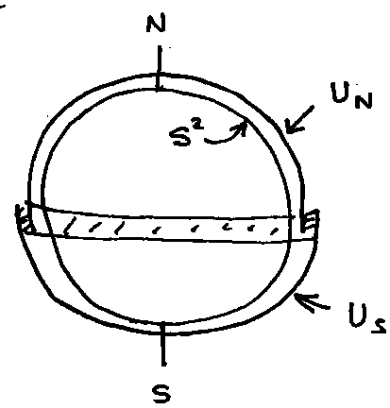
It has a string on the positive z-axis.



We see that it is necessary to cover $M = \mathbb{R}^3 - \{0\}$ with at least two charts, with $\vec{A}$ smooth in each chart. This suggests that the $U(1)$ P.F.B. for the charged particle in the monopole field is nontrivial (since a gauge convention somehow is related to a section of this PFB, and apparently this PFB does not have a global section).

The above is motivation to consider $U(1)$ bundles over $M = \mathbb{R}^3 - \{0\}$. Actually, it suffices to study $U(1)$ bundles over S^2, since the latter is a deformation retract of $\mathbb{R}^3 - \{0\}$. How many (inequivalent) circle bundles are there over S^2?

We can cover S^2 with two open sets, U_N which covers the northern hemisphere and extends slightly south of the equator, and U_S which does something similar from the southern side. These overlap slightly at the equator. If we have any bundle over S^2, then ~~this~~ the part of this bundle over U_N or U_S is trivial, since these are contractible. Thus there exist local trivializations ϕ_N and ϕ_S over these regions. The overlap $U_N \cap U_S$ is a small strip around the equator, where φ (the azimuthal angle) is a coordinate. The transition function $t_{NS} : U_N \cap U_S \to G = U(1)$ is therefore a function of φ, and we can write



$$t_{NS}(\varphi) = e^{i\alpha} \in U(1)$$

where $\alpha = \alpha(\varphi)$. We note that t_{NS}: equator $\to U(1)$, i.e., $t_{NS}: S^1 \to S^1$, so t_{NS} (as a map) belongs to some homotopy class in $\pi_1(S^1) = \mathbb{Z}$. The class is characterized by an integer $n \in \mathbb{Z}$.

5/4/04

Now consider a "local gauge transformation" defined by functions $g_N: U_N \to G$, $g_S: U_S \to G$, taking old transition functions to new ones according to

$$\tilde{\phi}_{N,x} = \phi_{N,x} \circ g_N(x)$$

$$\tilde{\phi}_{S,x} = \phi_{S,x} \circ g_S(x)$$

where $x \in U_N$ or U_S. The new transition functions are

$$\tilde{t}_{NS,x} = g_N(x)^{-1}\, t_{NS,x}\, g_S(x),$$

where now $x \in U_N \cap U_S$, i.e., $x \in$ equatorial strip. So we can replace x by φ in this formula if we want. To what extent can we simplify (or perhaps even transform away) the transition functions by such a gauge transformation? First note, all 3 factors above are elements of $U(1)$, an Abelian group, so we can write them in any order. Thus there is no loss of generality in letting $g_S(x) = e$ and letting $g_N(x)$ do all the work.

Look at U_N from above, where it is like a disk.

Examine the behavior of $g_N(x)$ as x goes around in a circle about the north pole. This gives a map: $S^1 \to U(1) = S^1$, which has a homotopy class $\in \pi_1(S^1) = \mathbb{Z}$. When the circle is very small, $g_N(x)$ is nearly constant, so the homotopy class is 0. By continuity, this must remain the homotopy class as the circle enlarges to the equator. Therefore on the equator, where we have

$$\tilde{t}_{NS}(\varphi) = g_N(\varphi)^{-1}\, t_{NS}(\varphi),$$

the homotopy class of $\tilde{t}_{NS}$ must be the same as that of t_{NS}.

5/4/04

We can change $t_{NS}(\varphi)$ into a ~~new~~ any function of the same homotopy class by means of a gauge transformation, but we cannot change it to any other homotopy class. In particular, if $n \neq 0$, we cannot gauge away t_{NS}, and the bundle is nontrivial.

We see that $U(1)$ bundles over S^2 are characterized by integers $\in \pi_1(S^1)$. There is an infinite but discrete set of such bundles.

The above analysis leaves two questions unanswered:

① How precisely is a choice of gauge for $\vec{A}$ linked to a choice of section in the PFB associated with the HLB carrying ψ for the charged particle?

② Since there are an ∞ number of $U(1)$ bundles over S^2, how do we know which is the right one for ψ?