

①  
4/29/04

$t_{ij,x}$  do, according to the definition given above. But this is just a convention (the way the  $t_{ij,x}$  were defined). There are various dictates in a standard presentation of fiber bundle theory about the direction (left, right) in which a group acts, but you will find in actual applications that you may want to reverse these.

The maps  $t_{ij,x} \in G$  can also be written in another notation,

$$t_{ij} : U_i \cap U_j \rightarrow G : x \mapsto t_{ij,x}.$$

The maps  $t_{ij}$  or  $t_{ij,x}$  are called transition functions. They specify the coordinate transformations on the fibers  $F_x$  in overlap regions. This is taking the point of view that the local trivializations provide coordinates on the fibers over the sets  $U_i$  (by labelling points of those fibers with points on the standard fiber). This is often a useful point of view.

---

We note that the transition functions satisfy the following properties:

$$(a) \quad t_{ii,x} = \text{id}_M \quad (\text{or } e \in G), \quad x \in U_i$$

$$(b) \quad t_{ij,x}^{-1} = t_{ji,x} \quad x \in U_i \cap U_j$$

$$(c) \quad t_{ij,x} t_{jk,x} = t_{ik,x} \quad x \in U_i \cap U_j \cap U_k.$$

These follow immediately from the definition,  $t_{ij,x} = \phi_{i,x}^{-1} \phi_{j,x}$ , and are not additional requirements. We use these later.

---

②  
4/29/04

The above is the definition of a coordinate bundle. It obviously has a lot of arbitrariness in it, due to the fact that there are huge numbers of possible open covers  $\{U_i\}$ , and likewise for the maps  $\phi_i$ . Suppose we have another coordinate bundle with the same  $E, M, F, G, \pi$  but set  $\{(V_i, \psi_i)\}$  in place of  $\{(U_i, \phi_i)\}$ . The new coordinate bundle, with  $\{(V_i, \psi_i)\}$ , satisfies all the requirements above of a coordinate bundle, in particular, the new transition functions  $\psi_{i,x}^{-1} \psi_{j,x} \in G$ . We will regard the new coord. bundle as compatible or equivalent to the previous bundle if the transition functions connecting the new and old open sets belong to the structure group,

$$\psi_{i,x}^{-1} \phi_{j,x} \in G, \quad \forall x \in V_i \cap U_j \\ \forall i, j.$$

Equivalently, we can throw the sets  $\{(U_i, \phi_i)\}$ ,  $\{(V_i, \psi_i)\}$  together (take their union), and require that the new set gives a coordinate bundle.  $\neq$

This compatibility condition is an equivalence relation, so the space of all coordinate bundles with given  $E, M, F, G, \pi$  but different  $\{(U_i, \phi_i)\}$  breaks up into equivalence classes. We call one such equivalence class a fiber bundle.

This definition of a fiber bundle obviously has a lot in common with the definition of a differentiable structure on a manifold (an equivalence class of atlases, each of which involves open ~~sets~~ covers  $\{U_i\}$  and maps  $\{\phi_i\}$  etc.).

③  
4/29/04

if some subset of the equivalence class of coordinate bundles that make up a fiber bundle involves transition functions that belong strictly to a subgroup  $H$  of the structure group  $G$ , then it may be desirable to redefine the structure group as  $H$  and to redefine the fiber bundle as the given subset of the original equivalence class (which becomes a new equivalence class w.r.t.  $H$ ). This is the reduction of the structure group. For example, normally we think of the structure group of the tangent bundle  $TM$  as  $GL(m, \mathbb{R})$ , but if we restrict consideration to orthonormal frames (on a Riemannian manifold) then we may wish to consider the structure group to be  $O(m)$ . For another example, most books say the structure group of the Möbius strip is  $\mathbb{Z}_2$ , because that is the smallest group necessary to get the essential twist. But you could consider the structure group to be a larger group (such as  $GL(1, \mathbb{R})$ ), if you take the fiber to be  $F = \mathbb{R}$ .

---

Suppose two coordinate bundles in the equivalence class making up a fiber bundle have the same open cover  $\{U_i\}$  of  $M$ , but different local trivializations, say,  $\{(U_i, \phi_i)\}$  and  $\{(U_i, \tilde{\phi}_i)\}$ . Then the compatibility condition ~~implies~~ says,

$$\phi_{i,x}^{-1} \tilde{\phi}_{j,x} \in G.$$

In particular, setting  $i=j$ , we get a  $G$ -valued function over  $U_i$ ,

$$\phi_{i,x}^{-1} \tilde{\phi}_{i,x} \equiv g_i(x) \in G,$$

$g_i: U_i \rightarrow G$ . Equivalently,

$$\tilde{\phi}_{i,x} = \phi_{i,x} g_i(x).$$

We may say, we apply  $g_i(x)$  to permute the points of the standard fiber  $F$  before applying  $\phi_{i,x}$  to get coordinates (the new,  $\sim$  coordinates) on actual fiber  $F_x$ . This transformation (from  $\phi$  to  $\tilde{\phi}$ ) can be regarded as a gauge transformation. A coordinate bundle with a specific  $\{(U_i, \phi_i)\}$  is equivalent to other coordinate bundles with the same  $\{U_i\}$  but with new  $\phi_i$  related by gauge transformations to the old one. The set of gauge transformations is quite large (the functions  $g_i: U_i \rightarrow G$  need only be smooth), so there are a lot of <sup>equivalent</sup> coordinate bundles with the same ~~set~~  $\{U_i\}$ . And of course there are a lot of ways of choosing an open cover  $\{U_i\}$  (although it must be sufficiently fine that local trivializations exist). This is a way of seeing that the equivalence class of coordinate bundles making up a fiber bundle is very large.

~~Suppose~~ (for a given coordinate bundle) it is possible to choose

Under a gauge transformation, the transition functions transform according to

$$\tilde{t}_{ij,x} = \tilde{\phi}_{i,x}^{-1} \tilde{\phi}_{j,x} = g_i(x)^{-1} t_{ij,x} g_j(x).$$

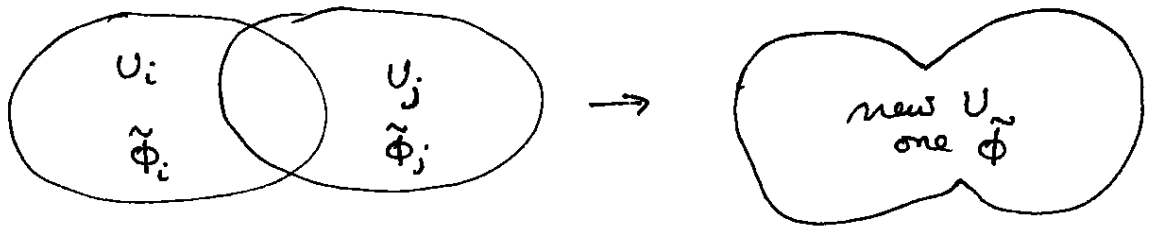
They are still elements of  $G$ , as they must be.

Suppose for a given coordinate bundle it is possible to choose  $g_i: U_i \rightarrow G$  and  $g_j: U_j \rightarrow G$  (for given  $i, j$ ) such that

$$\tilde{t}_{ij,x} = g_i(x)^{-1} t_{ij,x} g_j(x) = e,$$

$$\forall x \in U_i \cap U_j, \text{ i.e., } t_{ij,x} = g_i(x) g_j(x)^{-1}.$$

Then  $\tilde{\Phi}_{i,x} = \tilde{\Phi}_{j,x}$  on  $U_i \cap U_j$ , and we have "gauged away" the transition fns on the overlap region. Then we might as well combine regions  $U_i$  and  $U_j$  (take their union) and replace  $\tilde{\Phi}_i, \tilde{\Phi}_j$  by a new  $\tilde{\Phi}$  defined on the union:



If, for a given coordinate bundle, it is possible to gauge away all the transition functions in all the overlap regions, i.e., if it is possible to find gauge functions  $g_i: U_i \rightarrow G$  such that  $t_{ij,x} = g_i(x) g_j(x)^{-1}$ ,  $\forall x \in U_i \cap U_j$ ,  $\forall i, j$ , then we say that the fiber bundle (with this coord. bundle) in its equiv. class) is trivial. We may take this as the official definition of triviality.

In view of the remarks above, in the case of a trivial fiber bundle we can merge all the  $U_i$  together into a single  $U$ , which is their union. But this is  $M$  itself, since  $\{U_i\}$  is a cover of  $M$ . As for the corresponding local trivializations, it is now ~~a~~ a global trivialization,  $\Phi: M \times F \rightarrow \pi^{-1}(M) = E$ , that preserves fibers,  $\pi \Phi(x, f) = x$ . The converse of this is true, too. Thus, a fiber

4/29/04

is trivial iff it contains (its equivalence class contains) a coordinate bundle with just one  $U=M$ , and ~~one~~ one corresponding local (= global) trivialization, a fiber-preserving diffeomorphism

$$\phi: M \times F \rightarrow E,$$

$$\pi \phi(x, f) = x.$$

Thus we connect the official definition of triviality with the less formal remarks made previously.

Note that if you have just one  $(U, \phi)$  (for a trivial bundle), then there are no transition functions, and you might as well ~~reduce~~ reduce  $G$  to the trivial group  $\{e\}$ . You don't have to do this, but you can.

Now let's look at some examples of fiber bundles (previously discussed) and show that they actually are fiber bundles according to the official definition. Start with the tangent bundle. As for the spaces, we have  $E = TM = \bigcup_{x \in M} T_x M$ ,  $M = \text{same } M \text{ in both contexts}$ ,  $F = \mathbb{R}^m$ ,  $G = GL(m, \mathbb{R})$ . Projection  $\pi: TM \rightarrow M$  defined by  $\pi(T_x M) = x$ , so  $\pi^{-1}(x) = T_x M = F_x$  (fiber over  $x$ ). To complete the official defn, we need  $\{(U_i, \phi_i)\}$ . As for the open cover  $\{U_i\}$ , get this from an atlas on  $M$ . The atlas puts coordinates on the open sets  $U_i$ ; let coordinates on two of them ( $i$  and  $j$ ) be

$$x^\mu \text{ coords on } U_i, \quad e_\mu = \frac{\partial}{\partial x^\mu},$$

$$x'^\mu \text{ coords on } U_j, \quad e'_\mu = \frac{\partial}{\partial x'^\mu}.$$

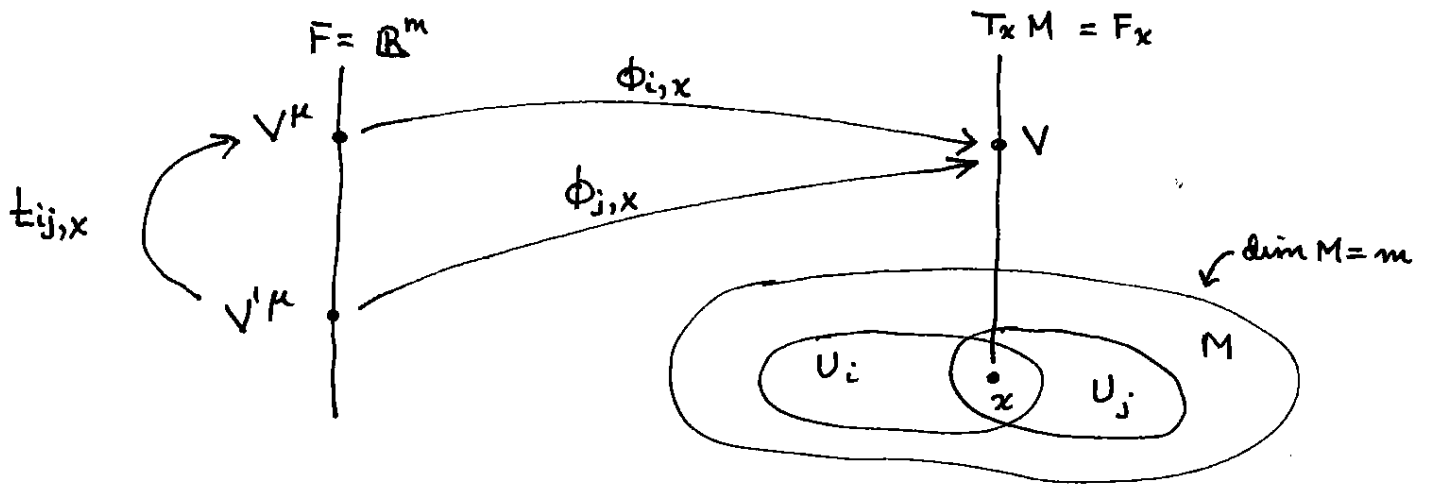
⑦  
4/29/04

Then we let the local trivializations be given by

$$\phi_{i,x} : \mathbb{R}^m \rightarrow T_x M : (v^1, \dots, v^m) \mapsto v^\mu e_\mu|_x, \quad (x \in U_i)$$

$$\phi_{j,x} : \mathbb{R}^m \rightarrow T_x M : (v'^1, \dots, v'^m) \mapsto v'^\mu e'_\mu|_x \quad (x \in U_j).$$

Now let  $x \in U_i \cap U_j$ , and think of  $v^\mu$  and  $v'^\mu$  as two sets of coordinates for one vector  $V \in T_x M$ . The map  $t_{ij,x}$  maps us from the  $j$ -coordinates to the  $i$ -coordinates ( $v'^\mu$  to  $v^\mu$ ):



By chain rule,

$$e'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} e_\nu,$$

or

$$e'_\mu = e_\nu J^\nu_\mu$$

where

$$J^\nu_\mu = \frac{\partial x^\nu}{\partial x'^\mu} = \text{Jacobian, } \det J \neq 0 \text{ so}$$

$$J \in GL(m, \mathbb{R}).$$

4/29/04

Thus we find

$$v^\mu = J^\mu_\nu \dot{v}^\nu,$$

the coordinates are related just by matrix multiplication (the obvious action of  $GL(m, \mathbb{R})$  on  $\mathbb{R}^m$ ). This is a left action, as required, and

$$t_{ij,x} = J(x) \in G, \quad x \in U_i \cap U_j.$$

Thus TM is a fiber bundle according to the official definition.

As an exercise you may show that if you use an <sup>compatible</sup> equivalent (but different) atlas, the resulting coordinate bundle is equivalent to the old one.

Next we consider the frame bundle. This is an example of a principal fiber bundle, so we define that first.

A principle fiber bundle is one for which the fiber is the same as the structure group,  $F=G$ . We also require the <sup>left</sup> action of  $G$  on  $F=G$  (required for the transition functions) be left multiplication. We write  $P$  instead of  $E$  for a principal fiber bundle. Since  $F=G$ , the list of spaces is  $(P, M, G)$ .

For the frame bundle we let  $F_x M$  be the set of all frames in  $T_x M$ , we define

$$FM = \bigcup_{x \in M} F_x M,$$

and we define  $\pi: FM \rightarrow M$  by  $\pi(F_x M) = x$ . We set  $P=M$ ,  $G=GL(m, \mathbb{R})$ , and use the same  $\pi$  for the bundle, so that  $\pi^{-1}(x) = F_x M = F_x$  (the fiber over  $x$  is the frame space over  $x$ ).



To create the required  $\{(U_i, \phi_i)\}$  we use an atlas to define the  $\{U_i\}$  and put coordinates  $x^\mu$  and  $x'^\mu$ , and frames  $e^\mu = \partial/\partial x^\mu$  and  $e'_\mu = \partial/\partial x'^\mu$ , on open sets  $U_i$  and  $U_j$ , respectively, as above with the tangent bundle.

The maps  $\phi_i, \phi_j$  are defined by

~~$$\phi_i : GL(m, \mathbb{R}) \rightarrow F_x M : A \mapsto \{f_\mu\}, \quad (x \in U_i)$$~~

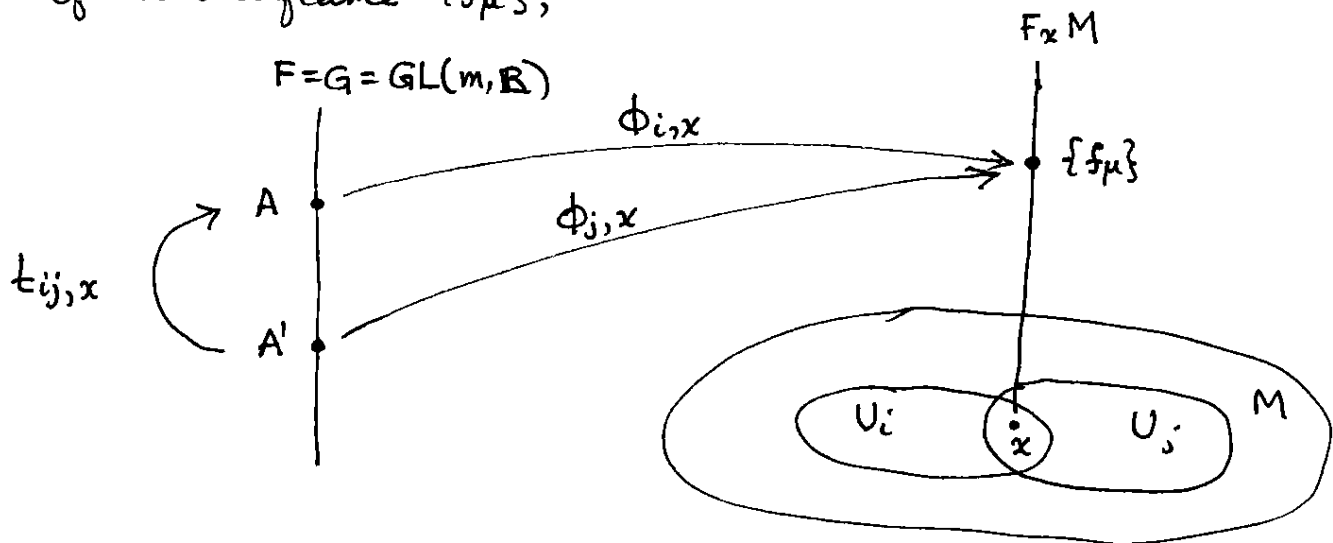
$$\phi_{i,x} : GL(m, \mathbb{R}) \rightarrow F_x M : A \mapsto \{f_\mu\}, \quad (x \in U_i)$$

$$\text{where } f_\mu = e_\nu A^\nu{}_\mu$$

$$\text{and } \phi_{j,x} : GL(m, \mathbb{R}) \rightarrow F_x M : A' \mapsto \{f'_\mu\}, \quad (x \in U_j)$$

$$\text{where } f'_\mu = e'_\nu A'^\nu{}_\mu$$

Here  $A, A'$  are matrices  $\in GL(m, \mathbb{R})$ . If we assume  $x \in U_i \cap U_j$  and set  $f_\mu = f'_\mu$  in the above, so that  $A, A'$  are the two coordinates of the one frame  $\{f_\mu\}$ ,



then

$$f_\mu = e_\nu A^\nu{}_\mu = e'_\nu A'^\nu{}_\mu = e_\sigma J^\sigma{}_\nu A'^\nu{}_\mu$$

where  $J$  is the Jacobian as before. In matrix language

this is

$$A = JA,$$

or

$$t_{ij,x} = J(x) \in GL(m, \mathbb{R}).$$

So the frame bundle  
is a P.F.B.



These are the same transition functions as in the tangent bundle.

They act on  $F=G$  by left multiplication, as required of a P.F.B.

The space of fiber bundles is divided into the principal fiber bundles and everything else. Vector bundles are included in the everything else. Principal fiber bundles have properties not shared by other bundles. In particular the structure group has an action on P in the case of a P.F.B. We will require this action to be a right action, in order to conform with the conventions established above. In a particular application, you may find it more convenient to think of the action of  $G$  on  $P$  as a left action, in which case you should revise all the definitions above so that  $t_{ij,x}$  are right actions.

For a general fiber bundle (not principal)  $G$  in general has no natural action on  $E$ . For example, in the tangent bundle  $G = GL(m, \mathbb{R})$  and  $T_x M$  is a vector space, but we cannot specify an action of  $G$  on  $T_x M$  until a basis is chosen in  $T_x M$ . In fact, such bases are selected when we set up the local trivializations, but these are only defined over the  $U_i$ , they don't usually agree in the overlaps, and they depend on the choice of the  $U_i$  and  $\phi_i$ . Thus there is no natural action of  $G$  on  $E = TM$ .

On a P.F.B., however,  $G$  always has a natural action. In specific examples, this is usually easy to see. For example, in the frame bundle  $FM$  choose a particular frame  $\{f_\mu\} \in F_x M$ . Let  $G = GL(m, \mathbb{R})$  act on this by

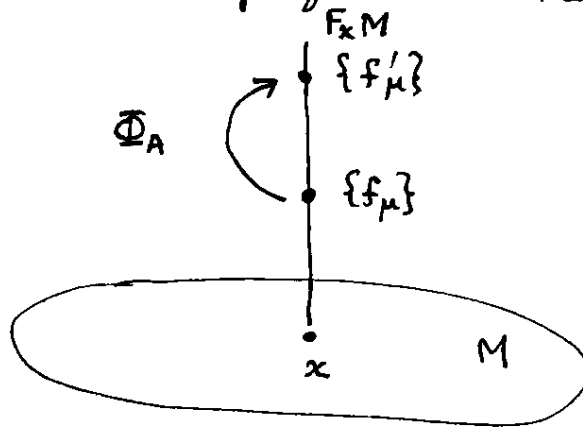
$$\Phi_A \bullet \{f_\mu\} = \{f'_\mu\}$$

$$\text{where } f'_\mu = f_\nu A^\nu{}_\mu,$$

where we call the action  $A \mapsto \Phi_A$ ,  $A \in GL(m, \mathbb{R})$  and  $\Phi_A: P \rightarrow P$  (here  $\Phi_A: FM \rightarrow FM$ ). Notice that

$$\Phi_B \Phi_A = \Phi_{AB},$$

so this is a right action. Note the following things about this action. First, it did not require a local trivialization for its definition; it is global. No coordinates on  $M$  or on  $F_x M$  are needed. Second, it maps fibers into themselves,



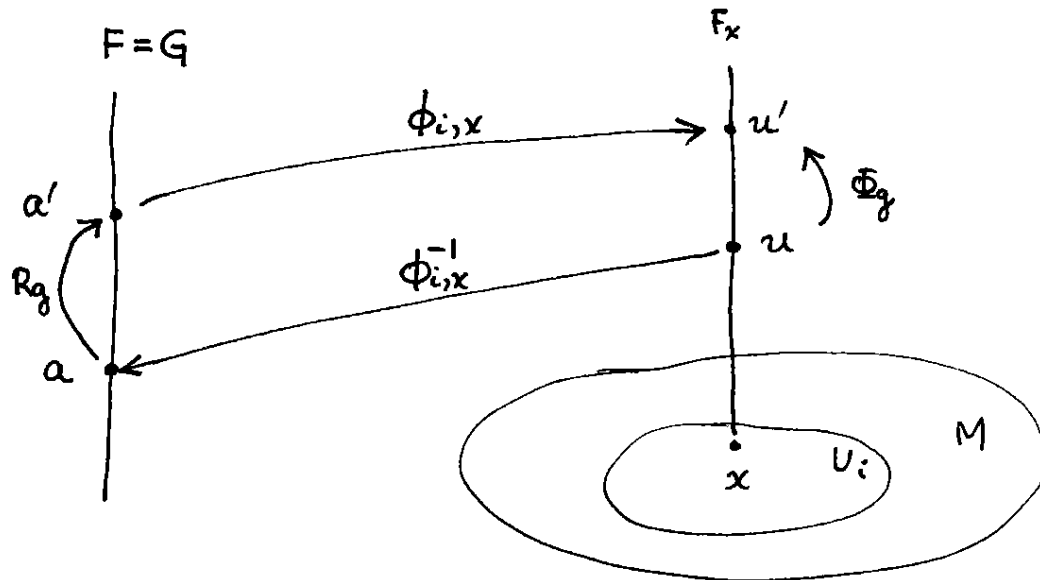
On fact, the orbit of the action is the entire fiber (the action is transitive on the fiber). Third, the action on  $P$  is free ( $\Phi_A \{f_\mu\} = \{f_\mu\}$  iff  $A = \text{Id}$ ).

A common way of creating a P.F.B. is to allow a group  $G$  to act on a manifold.  $\leftarrow$  call it  $P$ . If the action is free, then the orbits of the group action are diffeomorphic to  $G$ , and we can regard them as the fibers of a bundle. We assume the action of  $G$  on  $P$  is from the right. We then define  $M = P/G$ , which also defines  $\pi$ .

The Hopf fibration is an example of such a P.F.B. It arises from letting  $U(1)$  act on  $S^3$  (as specified above). So is the construction of ~~homogeneous~~ <sup>symmetric</sup> spaces mentioned above. ( ~~$P = G$~~ ) Here  $P = G$  (not the structure group),  $H$  is a subgroup of  $G$  ( $H$  is the structure group),  ~~$H$~~   $H$  acts on  $G$  by right multiplication ( $g \mapsto gh$ ), the orbits of the action are left cosets  $gH$ ,  $M = G/H$  is the symmetric space, the space of left cosets, and  $\pi: G \rightarrow M$ . In all these cases the action of the structure group (call it  $G$  again) on  $P$  is obvious: it is just the action that was used to form the bundle.

We now define the action of  $G$  on  $P$  in the general case, let  $g \in G$ . We wish to define  $\Phi_g: P \rightarrow P$ . Let  $u \in P$ .  $u$  belongs to some fiber  $F_x$ ,  $\pi(u) = x$ , and  $x$  lies in some  $U_i$ , speaking of a ~~set~~ set  $\{(U_i, \phi_i)\}$ . For  $x \in U_i$ , we define the action of  $\Phi_g$  as in the picture,

4/29/04



that is,

$$u' = \Phi_g u = \phi_{i,x} R_g \phi_{i,x}^{-1} u,$$

or 
$$\Phi_g u = \phi_{i,x} \left( (\phi_{i,x}^{-1} u) g \right).$$

This mapping is apparently only defined over  $U_i$ , and it apparently depends on the local trivialization. But suppose  $x \in U_i \cap U_j$ . Then

$$\phi_{i,x} = \phi_{j,x} \circ \tau_{ij,x}^{-1}, \text{ so}$$

$$\Phi_g u = \phi_{j,x} \left[ \tau_{ij,x}^{-1} \left( (\tau_{ij,x} \circ \phi_{j,x}^{-1} u) g \right) \right],$$

or by rearranging parentheses,

$$\Phi_g u = \phi_{j,x} \left( (\phi_{j,x}^{-1} u) g \right).$$

The answer does not depend on which local triv. we use, and in fact it is globally defined. Notice that  $\Phi_g$  preserves fibers, the orbits are the fibers, the action is free on  $\mathbb{P}$  and transitive on each

4/29/04

fiber, and it is a right action.

Here is another special property of a P.F.B., not shared by other fiber bundles.

Thm. A P.F.B. is trivial iff it possesses a global section.

Proof: (a). Suppose  $(P, M, G, \pi)$  is trivial. Then there exists

$\phi: M \times G \rightarrow P$  such that  $\pi \phi(x, g) = x$ . Define  $S: M \rightarrow P$  by

$$S(x) = \phi(x, e).$$

Here we can use any constant group element,  $e$  is just convenient. Then  $\pi S(x) = \pi \phi(x, e) = x$ , so  $S(x)$  is indeed a global section.

(b) Suppose  $\exists S: M \rightarrow P$  such that  $\pi(S(x)) = x$ . Then define  $\phi: M \times G \rightarrow P$  by

$$\phi(x, g) = S(x)g,$$

where we are using the right multiplication defined for any P.F.B. Then

$$\pi \phi(x, g) = \pi(S(x)g) = \pi(S(x)) = x,$$

where we use the fact that  $S(x)$  and  $S(x)g$  belong to the same fiber (right action is fiber preserving). ~~Thus, it is a fiber~~  
 Note also that  $\phi_x: G \rightarrow F_x: g \mapsto \phi(x, g) = S(x)g$  is a diffeomorphism, since the orbit of the right action is the whole fiber. (More exactly, the argument shows that  $\phi_x$  is invertible, the fact that it is a diffeomorphism follows from the general assumption of smoothness).

Thus,  $\phi$  is a diffeomorphism, and  $P$  is trivial. <sup>(the bundle)</sup>

(15)  
4/29/04

As noted previously, no similar theorem holds for other types of bundles. For example, every vector bundle has a global section (the zero section) whether or not it is trivial. But here is a useful theorem regarding vector bundles.

Thm. A vector bundle is trivial iff the corresponding frame bundle is trivial.

We will prove this for the special case of the tangent bundle  $TM$  and the frame bundle  $FM$  (which means the bundle of frames in the tangent spaces. Every ~~frame~~ vector bundle has a corresponding frame bundle.)

(a) Suppose  $FM$  is trivial. Then by the previous theorem there exists a field of frames  $\{f_\mu\}$ , globally defined and smooth everywhere. Then define

$$\begin{aligned} \phi: M \times \mathbb{R}^m &\rightarrow TM \\ &: (x, (v^1, \dots, v^m)) \mapsto v^\mu f_\mu|_x. \end{aligned}$$

This is a bijection hence a diffeomorphism, and fiber preserving,  $\pi \phi(x, \cdot) = x$ , so  $TM$  is trivial.

(b) Suppose  $TM$  is trivial. Then  $\exists \phi: M \times \mathbb{R}^m \rightarrow TM$ , a diffeo. such that  $\pi \phi(x, \cdot) = x$ . Let  $\{E_1, \dots, E_m\}$  be a basis in  $\mathbb{R}^m$  (each  $E_\mu$  is an  $m$ -vector of numbers, maybe the "unit vectors" in  $\mathbb{R}^m$ ). Map these onto  $TM$  using  $\phi$ , i.e.,

4/29/04

define

$$e_{\mu}|_x = \phi(x, E_{\mu}).$$

Then we get a frame in each tangent space, hence a field of frames, hence a global section of FM, hence FM is trivial.