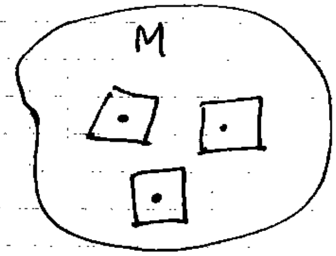
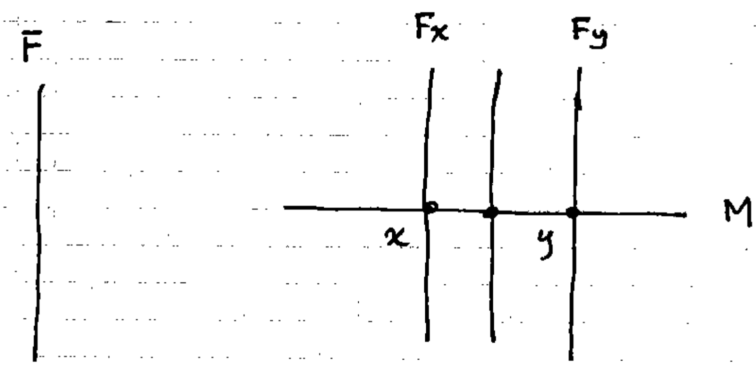
We turn now to fiber bundles. Begin with the intuitive idea, which is that a fiber bundle is a space made up by attaching identical copies of one space (the standard fiber) to each point of another space (the base space). For example, the tangent bundle TM to a manifold M is made up by attaching identical copies of $\mathbb{R}^m$ ($m = \dim M$) to each point $x \in M$. The copies are the tangent spaces $T_x M$.



Here M = base space

std fiber $= \mathbb{R}^m = F$.

Sometimes we sketch a fiber bundle as if F (the std fiber) and M (the base space) were one-dimensional, since it's hard to draw higher dimensions.

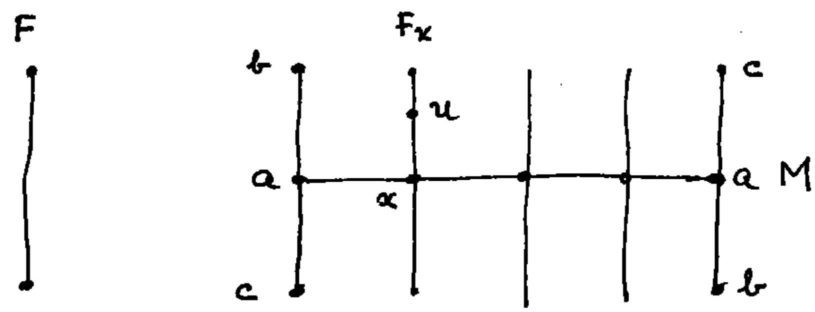


(M is the base space.)

Here F is the standard fiber, while F_x is the fiber over x, $x \in M$. F_x is required to be diffeomorphic to F (this is what we meant by "identical* copy.")

A fiber bundle is a way of creating a new space out of old spaces (here M and F). It is a generalization of the cartesian product. In fact, $M \times F$ is a fiber bundle (it is a way of attaching copies of F to points of M), but most fiber bundles are not, (i.e., not diffeomorphic to) $M \times F$. All fiber bundles, are, however, locally diffeo. to such a Cartesian product. That is, if $U \subset M$ is a sufficiently small open subset of M, then the set of fibers over U is diffeo. to $U \times F$. (This is sort of like the idea that a sufficiently small region of a differentiable manifold is diffeo. to a region of $\mathbb{R}^n$, the coordinate space.) The reason most fiber bundles are not globally diffeo. to $M \times F$ is that the different patches are fitted together with a kind of "twist".

The example of the Möbius strip will make this clear. The Möbius strip is the only nontrivial fiber bundle that is easy to visualize as a subset of $\mathbb{R}^3$. Here $M = S^1$ (the circle) and $F = [-1, 1]$, a closed interval. Denote M by a line segment aa with ends identified.

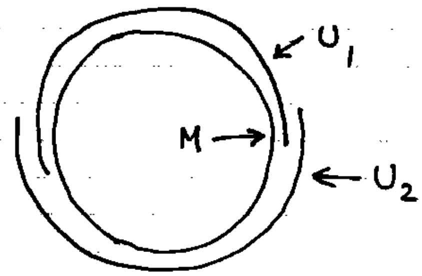


Because of the twist in the Möbius strip, the two fibers bc, cb over a are identified upside down.

The entire space, the set of all fibers over all $x \in M$, is denoted E (the entire space), or the bundle for short. If $u \in E$ is some point, it must belong to some fiber F_x over a point $x \in M$. Thus we define the projection map,

$$\pi : E \to M : u \mapsto x.$$

The set of fibers over $U \subset M$ is $\pi^{-1}(U)$. If U is small enough, $\pi^{-1}(U) \cong U \times F$ ($\cong$ means "is diffeomorphic to"). In the case of the Möbius strip, U need only be any proper subset of M. Let us cover $M = S^1$ with two open intervals, as illustrated:

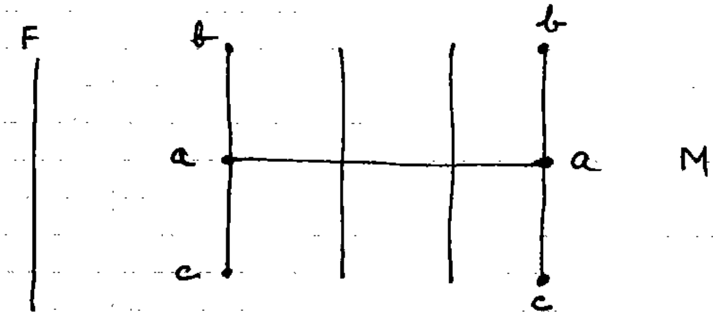


$U_1 \cup U_2 = M$.

Over each of U_1, U_2, the Möbius strip looks like a Cartesian product, but a twist is introduced in the overlaps that gives E a twisted product structure.

In general, a fiber bundle is said to be <u>trivial</u> iff $E \cong M \times F$. Triviality is a topological designation.

In the case of the Möbius strip, we could leave out the twist and we would get a cylinder:



In this case, $Cyl = S^1 \times F$ (the bundle is trivial). The Möbius strip and cylinder illustrate the fact that two bundles can have the same base space and same fibers but not be identical. In fact, a problem in fiber bundle theory is to classify all bundles with a given

M and F.

Now consider the <u>tangent bundle</u>. Some of this material was covered before. The tangent bundle TM has M = base space, F = standard fiber $= \mathbb{R}^m$, E = entire space = TM. The tangent bundle is defined by

$$TM = \bigcup_{x \in M} T_x M.$$

It is the collection of all tangent vectors attached to all possible points $x \in M$. The projection map $\pi : TM \to M$ is defined by $\pi(T_x M) = x$.

Before investigating the fiber bundle aspects of TM let's first prove that it is actually a differentiable manifold. (It is defined above as a collection of objects.) To do this, we must exhibit coordinates on TM. We are assuming M is a diff. manifold, so it possesses an open cover $\{U_i\}$ with associated charts and coordinates. Let x^μ be the coordinates on $U \in \{U_i\}$. Then $\{e_\mu = \partial/\partial x^\mu\}$ is a frame at each $x \in U$, so any $V \in T_x M$ can be written

$$V = V^\mu e_\mu|_x,$$

and we can take (x^μ, V^μ) as coordinates on $\pi^{-1}(U)$. ~~Thus~~ Notice that $\{\pi^{-1}(U_i)\}$ makes an open cover of TM, so coordinates like the above make an atlas on TM. TM is a differentiable manifold of dimension 2m. In (x^μ, V^μ), the x^μ tell you which fiber you are on, and the V^μ tell you where you are on the fiber.

The notion of a <u>vector field</u> can be given an interpretation in terms of the bundle TM. A vector field is of course a smooth assignment of a vector to each point $x \in M$. We can define it as a map,

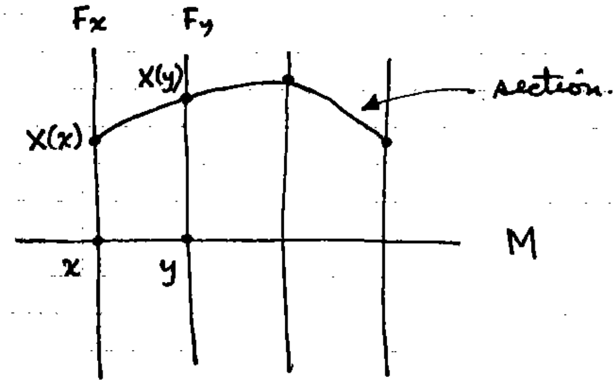
$$X : M \to TM$$

with the property,

$$\pi\left(X(x)\right) = x, \qquad \forall\, x \in M,$$

which guarantees that $X(x)$ is actually attached to x.

We can visualize a vector field geometrically as a submanifold of TM. If we just plot the points $X(x)$ in TM for $x \in M$, we get a surface:



The surface is called a (global) section of the fiber bundle. Exactly the same picture works for any fiber bundle, a global section is a ~~map: M→E~~ map $S: M \to E$ such that $\pi(S(x)) = x$, or rather, the section is the image of this map. The global section is a ~~surface~~ submanifold of TM, diffeomorphic to M, which intersects each fiber in one point.

The above is a global section. One can also discuss local sections, which are only defined over some subsets of M. For some bundles, global sections do not exist.

Similar to the tangent bundle TM is the cotangent bundle T^*M. It is the union of all the cotangent spaces $T^*_x M$. Similarly we can define the bundle of all type $(0,2)$ tensor fields, etc. etc. There is a bundle for each type of tensor field. These are all examples of vector bundles, i.e., bundles in which the standard fiber is a vector space.

Another type of vector bundle occurs when M is a submanifold of Euclidean $\mathbb{R}^n$ $(m = \dim M < n)$. Consider $x \in M$ and the

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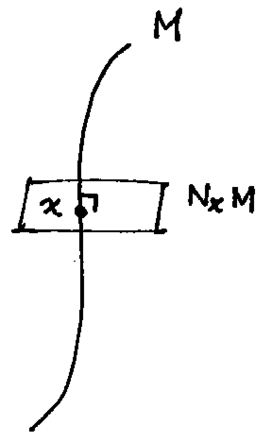
tangent space $T_x M$. This is a subspace of $T_x(\mathbb{R}^n)$. Define $N_x M$ as the normal space, i.e., the subspace of $T_x(\mathbb{R}^n)$ complementary to and orthogonal to $T_x M$, so that

$$T_x(\mathbb{R}^n) = T_x M \oplus N_x M$$

$$\text{and} \quad T_x M \perp N_x M.$$

For example, if M is the world line of a particle in Minkowski space-time, $\mathbb{R}^4$, then the normal space $N_x M$ is the 3D space



← Here $M = 1D$
$F = \mathbb{R}^3$
$E = NM$

← This normal bundle is important for Thomas precession.

of purely ~~spacelike~~ spatial vectors in the rest frame of the particle at x. The ~~fiber~~ normal bundle is then defined by

$$NM = \bigcup_{x \in M} N_x M.$$

~~this is~~ Another bundle of interest is the frame bundle to a manifold M. Let $x \in M$, and let $\{e_\mu\}$ be a frame at x (a ~~set of vec~~ basis in $T_x M$). We wish to construct a fiber F_x that will consist of all frames at x. If $\{f_\mu\}$ is another frame, it is related to $\{e_\mu\}$ by a nonsingular matrix, (a real, $m \times m$ matrix)

$$f_\mu = e_\nu A^\nu{}_\mu, \quad \text{where } \det(A) \neq 0.$$

That is, $A \in GL(m, \mathbb{R})$. Thus (once the reference frame $\{e_\mu\}$ is chosen) frames in F_x are placed in one-to-one correspondence with elements of $G = GL(m, \mathbb{R})$, and the standard fiber is $F = G = GL(m, \mathbb{R})$. Then the frame bundle FM is defined by

$$FM = \bigcup_{x \in M} F_x ,$$

and $\pi : FM \to M$ is defined in the usual way.

Frame bundles do not in general possess global sections. We say that M is <u>parallelizable</u> if FM possesses a global section. This would mean that we could construct m vector fields $\{e_\mu\}$ on M that were everywhere linearly independent. In particular, none of the e_μ could vanish anywhere.

A theorem states that the frame bundle FM is trivial iff it possesses a global section. Another theorem states that if FM is trivial, then so is TM (and T^*M and all the other tensor bundles). These are easy to prove, but we'll wait until we have proper definitions.

The frame bundle is an example of a <u>principle fiber bundle</u>. A principle fiber bundle is one in which $F = G =$ the structure group (defined later).

A vector bundle always possesses a global section, it is just the <u>zero-section</u>, for which the vector 0 is assigned to each $x \in M$. Every vector space has a zero, so this is meaningful.

A more challenging question is to ask if a vector bundle possesses a section that vanishes nowhere. Let's talk about TM and vector field. A zero of a vector field, a point x where $X(x)=0$, is usually regarded as a <u>singularity</u> of the vector field. X is smooth there (we are assuming we have only smooth fields), but the direction of X is not defined. Some manifolds do not possess a global, nonsingular (= nonzero) section of TM.

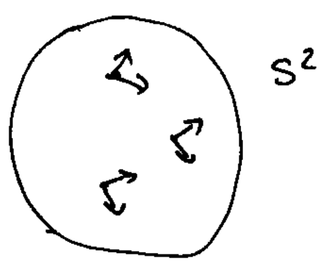
Here is a theorem (Poincaré-Hopf). It is discussed in Frankel but not Nakahara.

<u>Theorem</u>: TM possesses a global, nonvanishing (nowhere vanishing) section iff $\chi(M)=0$, where χ is the Euler characteristic.

For example, $\chi(S^2)=2$, so there does not exist on S^2 any smooth vector field that vanishes nowhere. This is the "hair on the coconut" theorem. But on the torus, $\chi(T^2)=0$, and such vector fields exist.

Since the vector fields that make up a ~~frame~~ global section of the frame bundle vanish nowhere, we see that if $\chi(M)\neq 0$ then the frame bundle is nontrivial, and has no global sections. For example, it is impossible to construct a smooth field of frames on S^2.

must have a discontinuity somewhere. →



Such fields of frames are used in optics for polarization vectors of light waves. $\vec{k}$ is the radial vector (in $\vec{k}$-space), and $\hat{\varepsilon}_1, \hat{\varepsilon}_2$ are unit vectors (the polarization vectors) that span the plane $\perp$ to $\vec{k}$. The books seldom mention that $\hat{\varepsilon}_1, \hat{\varepsilon}_2$ cannot be assigned smoothly over the sphere.

In the above examples we constructed a fiber bundle by starting with the base space and adding fibers at each point. In the next example we start with E and work down to M.

Let $E = S^3$. We visualize $S^3 \subset \mathbb{R}^4$ as the set of unit vectors in $\mathbb{R}^4 \cong \mathbb{C}^2$. Let (x_1, x_2, x_3, x_4) be coordinates on $\mathbb{R}^4$, let

$$z_1 = x_1 + i x_2$$
$$z_2 = x_3 + i x_4$$

let

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2,$$

so that

$$|z|^2 = |z_1|^2 + |z_2|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

Think of z as a 2-component spinor. Thus, the points of S^3 can be thought of as normalized spinors. In the following assume $|z|^2 = 1$, so $z \in S^3$.

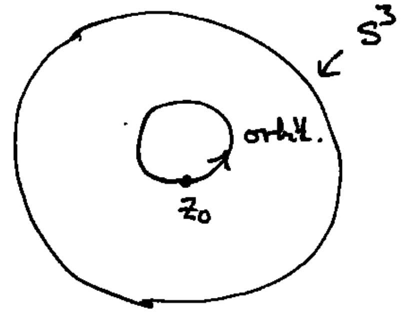
We define an action of $U(1)$ on S^3 by

$$z \mapsto e^{i\alpha} z, \quad e^{i\alpha} \in U(1), \qquad 0 \le \alpha < 2\pi.$$

We are changing the "overall phase" of the spinor. This action is <u>free</u>, which means that the isotropy subgroup is trivial $\{e\}$ (that is, if $z = e^{i\alpha} z$, then $\alpha = 0$). Thus the orbit of the $U(1)$ action is a circle (diffeo. to $U(1)$), and we have a foliation of S^3 into a 2D family of circles.

Now we define

$$M = \frac{S^3}{U(1)},$$



that is, M is the quotient space of this group action. The projection map $\pi: S^3 \to M$ is defined by $\pi(z) = \pi(z')$ iff $z' = e^{i\alpha} z$, that is, π maps an entire orbit in S^3 onto a point of M. Thus, the orbit is the fiber F_x over $x \in M$; x is a label of the orbit. The standard fiber is $F = U(1) \cong S^1$. This is another example of a principle fiber bundle.

What is the space M? Suppose we wished to find coordinates on M. These would be scalar fields on M, say, $f: M \to \mathbb{R}$. Such scalar fields could be lifted to S^3 by the pullback, $\pi^* f: S^3 \to \mathbb{R}$. Such scalars $\pi^* f$ have the property that they are constant on the orbits of the U(1) action. Conversely, any scalar $: S^3 \to \mathbb{R}$ const. on the U(1) orbits can be projected to a scalar $: M \to \mathbb{R}$ using π.

A scalar on S^3 that is const. on the U(1) orbits must have the same value at any two points z and $z' = e^{i\alpha} z$. An obvious way to create such scalars is to use bilinear quantities, $(z^\dagger A z) = \langle z | A | z \rangle$ in spinor language, where A is a 2×2 matrix.

Let us define $H: S^3 \to \mathbb{R}^3$ by

$$H(z) = \langle z | \vec{\sigma} | z \rangle = \vec{n}(z),$$

where $\vec{\sigma}$ are the Pauli matrices, and $\vec{n}(z)$ is notation for the value of the function. $\vec{n}(z)$ is a real vector because $\vec{\sigma}$ is Hermitian, and of course $\vec{n}(z)$ is constant on $U(1)$ orbits. Then we have two simple theorems:

(a) $\vec{n}(z)$ is a unit vector, $|\vec{n}(z)|=1$. So write $\hat{n}(z)$ instead, and regard the map H as

$$H: S^3 \to S^2$$

where $\hat{n}(z)$ indicates a point on $S^2 \subset \mathbb{R}^3$.

(b) $\hat{n}(z) = \hat{n}(z')$ iff $z' = e^{i\alpha} z$. This part shows that orbits in S^3 are placed in one-to-one corresp. with points of S^2. Thus, $M = S^2$, and we have

$$\frac{S^3}{U(1)} = S^2.$$

This is called the <u>Hopf fibration</u>, and H is the <u>Hopf map</u>. It gives us an example of a circle bundle ($U(1) \cong S^1$) over S^2.

Another example. Let G = any Lie group, and H = a Lie subgroup. Let G/H = the space of (left, say) cosets. Then we have the structure of a (principle) fiber bundle, in which $E = G$, $F = H$, $M = G/H$, $\pi: G \to G/H: g \mapsto [g] = gH$.

In an early homework it was shown that

$$\frac{SO(3)}{SO(2)} = S^2,$$

where $SO(2)$ is the subgroup of rotations about the z-axis (say).

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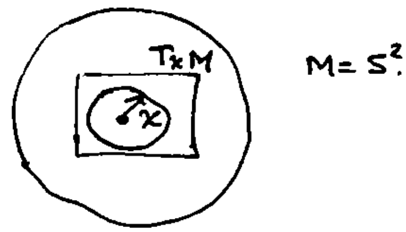
But $SO(2) \cong S^1$ and $SO(3) \cong \mathbb{R}P^3$, so we have again an example of a circle bundle over S^2, but this time the entire space $E = \mathbb{R}P^3$ (not S^3 as in the Hopf fibration). The difference is in the twistings applied to the way circles are attached to S^2.

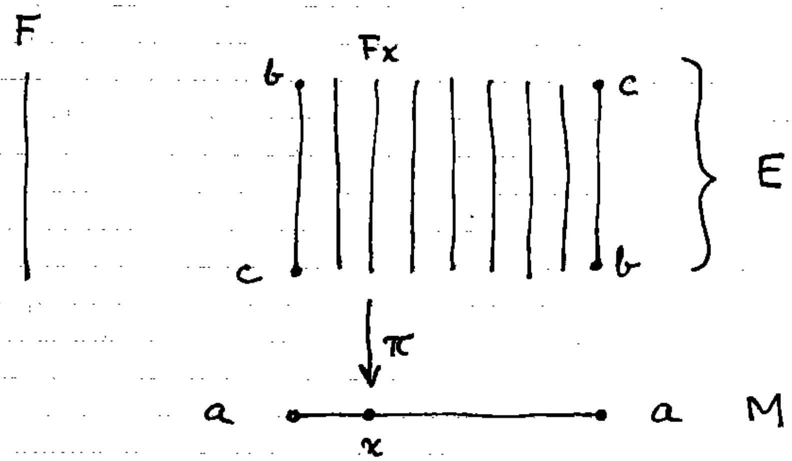
Another circle bundle over S^2 is easy to construct. Give S^2 the standard metric by embedding in $\mathbb{R}^3$, and consider the unit circle bundle, i.e., the set of vectors ~~xxx~~ X such that $|X| = 1$:



It is an exercise to see if this circle bundle over S^2 is different from the two previous ones.

Two things to note about these examples. First, note that in general there is no <u>natural</u> identification between the standard fiber F and the fibers F_x over points $x \in M$. For example, in the tangent bundle TM, each T_xM is certainly diffeo. to $\mathbb{R}^n$, but we get a specific identification of these two spaces only when we introduce a basis in T_xM. Similarly, in the frame bundle FM we obtain an identification of the fiber with $GL(n, \mathbb{R})$ only <u>after</u> a specific (reference) frame is chosen; then all other frames are related to this ~~one~~ by group actions. To say this another way, in the case of a principle fiber bundle (such as the frame bundle) each fiber is diffeo. to the structure group G, but unlike the group G itself, the fiber has no preferred "identity element."

Second, note that the base space ~~B~~ M should not properly be considered a submanifold of E, but rather a quotient space. This is clear in examples above such as the Hopf fibration, but it true also in other examples such as the Möbius strip. In fact, the drawings above of the Möbius strip as a fiber bundle were wrong, we should instead have drawn something like this:



In fact, it's easy to show that M can be identified with a submanifold of a bundle only when a global section exists (something we don't want to assume in general).

Now we turn to the official definition of a fiber bundle.

A coordinate bundle consists of the following.

1. Spaces:
   (a) M the base space
   (b) F the standard fiber
   (c) E the entire space (or "the bundle")
   + more later

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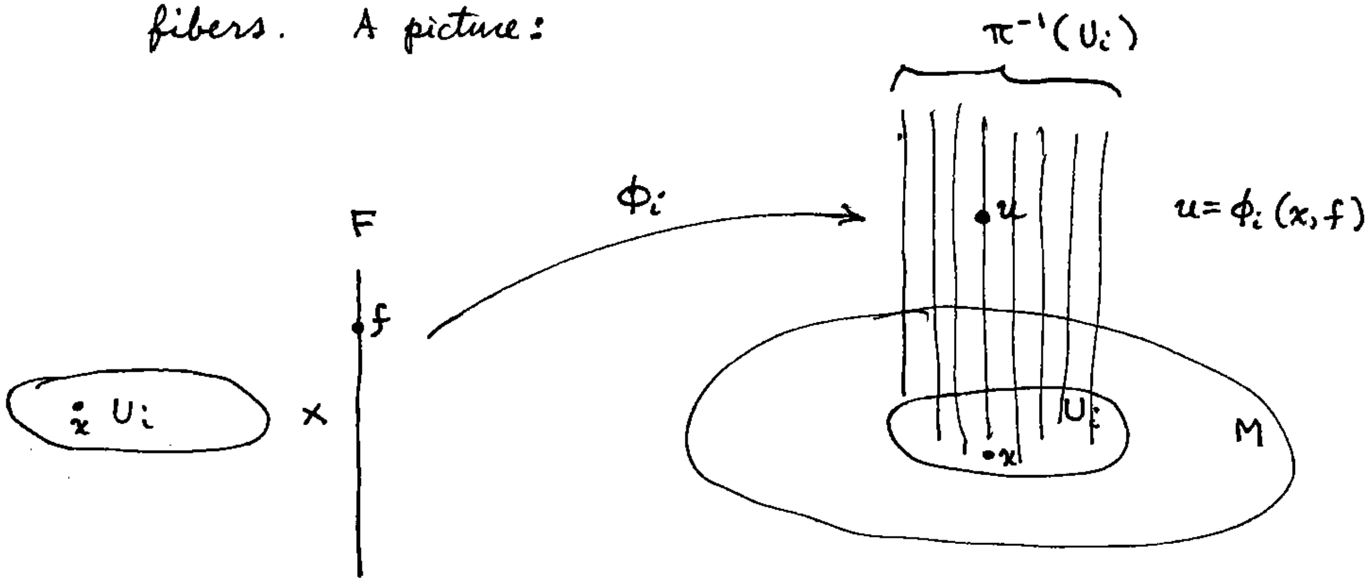
2. A surjective map, $\pi : E \to M$ (smooth, of course).
We define $F_x = \pi^{-1}(x)$ (for $x \in M$) as the <u>fiber over x</u>.
F_x is required to be diffeomorphic to the standard fiber F.

3. A set $\{(U_i, \phi_i)\}$, where $\{U_i\}$ is an open cover of M, and the ϕ_i are diffeomorphisms,

$$\phi_i : U_i \times F \to \pi^{-1}(U_i)$$

such that $\pi\, \phi_i(x, f) = x$ for $\forall\, x \in U_i$, $f \in F$. The ϕ_i are called <u>local trivializations</u>. These maps make precise the notion that a fiber bundle is locally a Cartesian product. The condition $\pi\, \phi_i(x,f) = x$ means that the maps preserve fibers. A picture:



It is useful to define ϕ_i restricted to one fiber (the one over x), by

$$\phi_{i,x} : F \to F_x,$$

$$\phi_{i,x}(f) = \phi_i(x, f).$$

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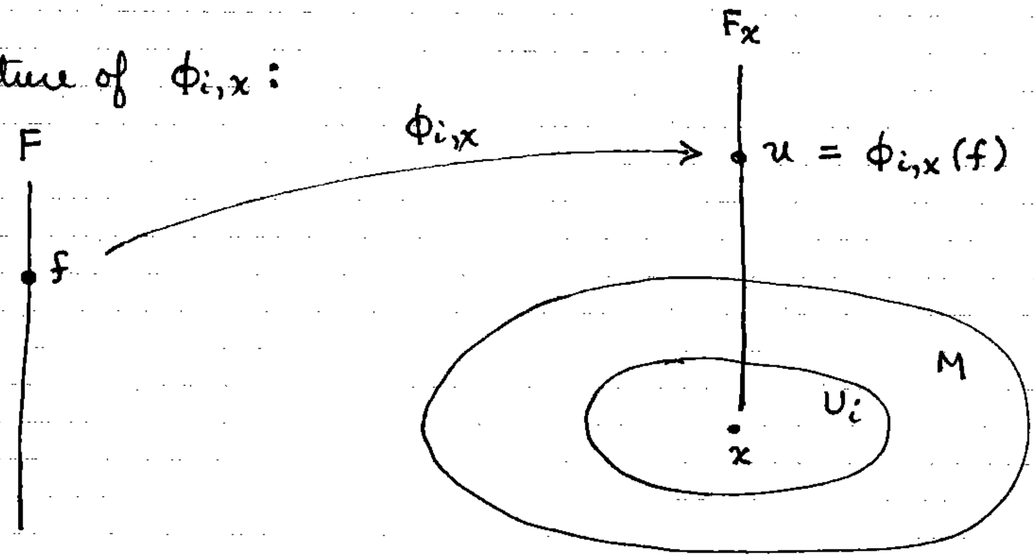
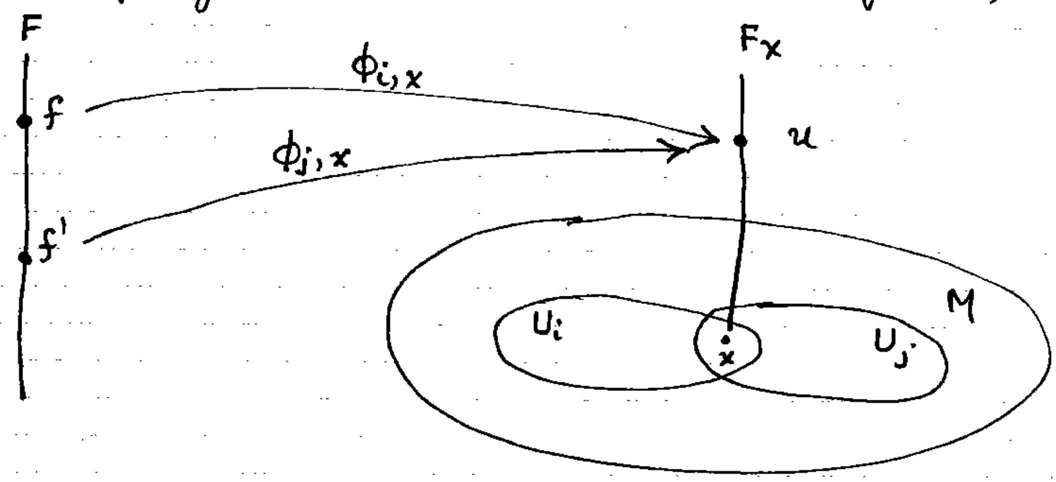
A picture of $\phi_{i,x}$:



We can think of $\phi_{i,x}$ as putting coordinates on the fiber F_x (labelling points on F_x by points f on the standard fiber.)

In an overlap region there will be two "coordinates systems",



The "coordinate transformation" is a map: $F \to F$, taking f' to f in the picture above. It is denoted $t_{ij,x} : F \to F$,

$$t_{ij,x} : F \to F \qquad (x \in U_i \cap U_j)$$

$$t_{ij,x} = \phi_{i,x}^{-1} \circ \phi_{j,x} \qquad (\text{defn of } t_{ij,x}).$$

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The maps $t_{ij,x}$ are diffeomorphisms $: F \to F$, and so belong to the ∞-dimensional group $\mathrm{Diff}(M)$. But in practice, the $t_{ij,x}$ usually belong to a much smaller subgroup of $\mathrm{Diff}(M)$, which we will call G. For example, in the tangent bundle, the standard fiber F is a vector space, and the $t_{ij,x}$ usually consist of linear transformations, so $G = GL(m, \mathbb{R})$ or perhaps a subgroup of $GL(m, \mathbb{R})$.

Thus we add one more space to the list under 1. above:

1. (d). G the <u>structure group</u>.

It would be possible to describe G as a subgroup of $\mathrm{Diff}(F)$, but sometimes we want to use the same G for different kinds of fibers, so we prefer to think of G as an abstract group with an action on F. ~~Thus we modify our~~ We require this action to be effective because that means that G is isomorphic to a subgroup of $\mathrm{Diff}(F)$. So we modify the above by saying,

4. The structure group G has an effective action on F. If $a \in G$ and $f \in F$, we write simply af instead of $\Phi_a f$ or some more complicated notation. G acts on F from the <u>left</u>. The maps $t_{ij,x} : F \to F$, which ~~change~~ map "j-coordinates" on F_x into "i-coordinates", coincide with the action of some element in G; thus we may equally well think of $t_{ij,x} \in G$.

We require G to act from the left because that's what the maps