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Summary:

$$\Omega = \sqrt{|g|}\ \theta^1 \wedge \ldots \wedge \theta^m = \frac{\sqrt{|g|}}{m!}\ \varepsilon_{\mu_1 \ldots \mu_m}\ \theta^{\mu_1} \wedge \ldots \wedge \theta^{\mu_m}$$

$$\Omega_{\mu_1 \ldots \mu_m} = \sqrt{|g|}\ \varepsilon_{\mu_1 \ldots \mu_m}$$

$$\Omega^{\mu_1 \ldots \mu_m} = \frac{\mathrm{sgn}(g)}{\sqrt{|g|}}\ \varepsilon_{\mu_1 \ldots \mu_m}$$

$$\Omega = *1$$

$$*: \Omega^r(M) \to \Omega^{m-r}(M)$$

$$*(\theta^{\mu_1} \wedge \ldots \wedge \theta^{\mu_r}) = \frac{1}{(m-r)!}\ \Omega^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_{m-r}} \left(\theta^{\nu_1} \wedge \ldots \wedge \theta^{\nu_{m-r}}\right)$$

$$\left.\begin{aligned} ** &= \mathrm{sgn}(g)(-1)^{r(m-r)} \\ *^{-1} &= \mathrm{sgn}(g)(-1)^{r(m-r)}\, * \end{aligned}\right\} \text{ on } \omega \in \Omega^r(M)$$

$$\alpha \wedge *\beta = \left(\frac{1}{r!}\ \alpha_{\mu_1 \ldots \mu_r}\, \beta^{\mu_1 \ldots \mu_r}\right) \Omega, \qquad \alpha, \beta \in \Omega^r(M)$$

$$\alpha \wedge *\beta = \beta \wedge * \alpha$$

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta = \langle \beta, \alpha \rangle \quad \text{(pos def. if } g \text{ pos def.)}$$

$$\langle \alpha, d\beta \rangle = \langle d^+\alpha, \beta \rangle, \qquad \forall\ \alpha \in \Omega^r(M),\ \beta \in \Omega^{r-1}(M).$$

$$d^+: \Omega^r(M) \to \Omega^{r-1}(M)$$

$$d: \Omega^r(M) \to \Omega^{r+1}(M)$$

$$d^+ = (-1)^r *^{-1} d * = \mathrm{sgn}(g)(-1)^{mr+m+1} * d *. \quad (\text{on } \omega \in \Omega^r(M))$$

$$d^+ d^+ = 0$$

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Now work out the action of d^{+} on a 1-form $\alpha \in \Omega^1(M)$. (We know d^{+} annihilates 0-forms). $d^{+}\alpha$ is a scalar, want to find it. Write $\alpha = \alpha_\mu \theta^\mu$.

First compute $*\alpha$:

$$*\alpha = \frac{\alpha_\mu}{(m-1)!} \Omega^{\mu}{}_{\nu_2 \dots \nu_m} (\theta^{\nu_2} \wedge \dots \wedge \theta^{\nu_m}) \qquad \text{(Raise + lower } \mu\text{)}$$

$$= \frac{1}{(m-1)!} \alpha^\mu \underbrace{\boxed{\Omega_{\mu\nu_2\dots\nu_m}}}_{\sqrt{|g|}\, \varepsilon_{\mu\nu_2\dots\nu_m}} (\theta^{\nu_2} \wedge \dots \wedge \theta^{\nu_m})$$

$$d*\alpha = \frac{1}{(m-1)!} \left(\sqrt{|g|}\, \alpha^\mu\right)_{,\sigma} \varepsilon_{\mu\nu_2\dots\nu_m} \underbrace{\theta^\sigma \wedge \theta^{\nu_2} \wedge \dots \wedge \theta^{\nu_m}}$$

$$= \varepsilon_{\sigma\nu_2\dots\nu_m} \underbrace{\theta^1 \wedge \dots \wedge \theta^m}_{\frac{\Omega}{\sqrt{|g|}}} .$$

$$d*\alpha = \frac{1}{(m-1)!} \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|}\, \alpha^\mu\right)_{,\sigma} (m-1)!\, \mathrm{sgn}\begin{pmatrix}\sigma\\ \mu\end{pmatrix} \Omega$$

Note: $\mathrm{sgn}\begin{pmatrix}\sigma\\ \mu\end{pmatrix} = \delta^\sigma_\mu$. More generally,

$$\mathrm{sgn}\begin{pmatrix}\sigma_1 \dots \sigma_r\\ \mu_1 \dots \mu_r\end{pmatrix} = \begin{vmatrix} \delta^{\sigma_1}_{\mu_1} & \dots & \delta^{\sigma_1}_{\mu_r} \\ \vdots & & \vdots \\ \delta^{\sigma_r}_{\mu_1} & \dots & \delta^{\sigma_r}_{\mu_r} \end{vmatrix}$$

$$= \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|}\, \alpha^\mu\right)_{,\mu} \Omega .$$

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A digression on a useful theorem. Let $X = X^\mu e_\mu$ be a vector field. Then

$$\boxed{X^\mu{}_{;\mu} = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}\, X^\mu\right)_{,\mu}.}$$

useful formula.

We used this theorem in computing Einstein field eqns. from Lagrangian. To prove it, expand RHS,

$$\text{RHS} = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}\right)_{,\mu} X^\mu + X^\mu{}_{,\mu}$$

But by the formula for the derivative of a determinant,

$$= \frac{1}{2}\left(g^{\alpha\beta} g_{\alpha\beta,\mu}\right) X^\mu + X^\mu{}_{,\mu}.$$

Now

$$X^\mu{}_{;\nu} = X^\mu{}_{,\nu} + \Gamma^\mu_{\alpha\nu} X^\alpha,$$

so

$$\text{LHS} = X^\mu{}_{;\mu} = X^\mu{}_{,\mu} + \Gamma^\mu_{\alpha\mu} X^\alpha.$$

Also,

$$\Gamma^\mu_{\alpha\beta} = \tfrac{1}{2} g^{\mu\nu}\left(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}\right),$$

so

$$\Gamma^\mu_{\alpha\mu} = \tfrac{1}{2} g^{\mu\nu}\left(\cancel{g_{\nu\alpha,\mu}} + g_{\nu\mu,\alpha} - \cancel{g_{\alpha\mu,\nu}}\right),$$

(two terms cancel by exchange $\mu \leftrightarrow \nu$ and symmetry)

$$\Gamma^\mu_{\alpha\mu} = \tfrac{1}{2} g^{\mu\nu} g_{\mu\nu,\alpha}.$$

So

$$\text{LHS} = X^\mu{}_{,\mu} + \tfrac{1}{2} g^{\mu\nu} g_{\mu\nu,\alpha} X^\alpha = \text{RHS}.$$

QED

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So to go back, we have

$$d{*}\alpha = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}\,\alpha^{\mu}\right)_{,\mu}\Omega = \alpha^{\mu}{}_{;\mu}\,\Omega$$

Now apply $(-1)^r *^{-1} = -*^{-1}$, note that $\Omega = *1$ so $*^{-1}\Omega = 1$, get

$$d^{+}\alpha = -*^{-1}d*\alpha = -\alpha^{\mu}{}_{;\mu}.$$

$$\boxed{d^{+}\alpha = -\alpha^{\mu}{}_{;\mu}}$$

It is minus the covariant ~~do~~ "divergence" of α (converted to a vector via g).

Note: in special case $\alpha = df$ $(f \in \Omega^0(M))$,
we have

$$d^{+}df = -f^{;\mu}{}_{;\mu}.$$

This is (minus) the obvious generalization of the Laplacian to curved spaces,

$$-\nabla^2 f = -f_{,ii} \quad \text{on Euclidean } \mathbb{R}^n.$$

~~[illegible]~~ Another note on this result:

$$\int_M d^n x\sqrt{|g|}\;\alpha^{\mu}{}_{;\mu} = -\int_M (d^{+}\alpha)\Omega = -\int_M d^{+}\alpha\wedge *1$$

$$= -\langle d^{+}\alpha, 1\rangle = -\langle\alpha, d1\rangle = 0.$$

A more straightforward way to see the same thing is to use integration by parts,

$$\int d^m x \sqrt{|g|}\ \alpha^\mu{}_{;\mu} = \int d^m x\ (\sqrt{|g|}\,\alpha^\mu)_{,\mu} = 0.$$

You have to convert a covariant deriv. to an ordinary deriv. if you want to integrate by parts.

Now Hodge * and Maxwell eqns (E+M). Already noted,

$$S_{EM} = \langle F, F\rangle = \int F \wedge * F.$$

Maxwell eqns in SR:

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \qquad \text{or} \qquad F = dA, \qquad F = \tfrac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

$$F_{[\mu\nu,\sigma]} = 0 \qquad \text{or} \qquad dF = 0$$

$$F^{\mu\nu}{}_{,\nu} = J^\mu$$

$$J^\mu{}_{,\mu} = 0$$

We use the comma goes to semicolon rule to put these into GR. For example,

$$F_{[\mu\nu;\sigma]} = 0.$$

Question: does this still mean $dF = 0$? (There are extra terms involving Γ from the covariant derivatives). Answer: Yes, because in the LC connection, all the Γ terms cancel when computing the components of an exterior derivative of any ~~scalar valued~~ (real-valued, i.e., not Lie algebra-valued) form. Thus,

$$F_{[\mu\nu;\sigma]} = 0 \Rightarrow F_{[\mu\nu,\sigma]} = 0 \Rightarrow dF \overset{=}{=} 0 \Rightarrow$$

$$\Rightarrow F = dA \quad (\text{Poincaré lemma}) \Rightarrow F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$$

$$\Rightarrow F = A_{\nu;\mu} - A_{\mu;\nu}.$$

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charge conservation

As for $J^{\mu}{}_{,\mu} = 0$ (in SR) it becomes $J^{\mu}{}_{;\mu} = 0$ (in GR)

Now define

$$J = J_\mu dx^\mu \qquad \text{(current 1-form)},$$

and charge conservation becomes

$$d^+ J = 0.$$

Finally, as for $F^{\mu\nu}{}_{,\nu} = J^\mu$ (in SR), it becomes $F^{\mu\nu}{}_{;\nu} = J^\mu$ (in GR). It can be shown (exercise for you) that this is equivalent to

$$d^+ F = J,$$

which is consistent with charge conservation because $d^+ J = d^+ d^+ F = 0$.

Summary of Maxwell eqns:

$$\boxed{\begin{array}{ll} F = dA, & d^+ J = 0. \\ dF = 0 & \\ d^+ F = J & \end{array}}$$

We can use these eqns to get a wave eqn. for A:

$$d^+ F = d^+ d A = J.$$

Although $d^+ d$ acting on scalars (we saw above) is (minus) the covariant Laplacian (i.e., d'Alembertian in space-time), this is not quite true for forms of higher rank ($r \geq 1$). For arb. forms we define,

$$\Delta = d^+ d + d d^+.$$

This agrees with the case of scalars since $d^+f=0$ (any scalar f), so

$$\Delta f = d^+ df.$$

But on a 1-form such as A we have

$$\Delta A = J - dd^+A.$$

The term on the RHS vanishes if we choose Lorentz gauge, $d^+A=0$.

[Think: $-\nabla^2\vec{A} = \vec{J} - \nabla(\nabla\cdot\vec{A})$ in NR magnetostatics.]

Now we explore the properties of the operator Δ.

$$\boxed{\begin{aligned}\Delta &= d^+d + dd^+ \\ &= (d+d^+)^2\end{aligned}} \quad \text{(defn)} \quad \text{since } d^2 = d^{+2} = 0.$$

Actually, to simplify the functional analysis it helps to assume that M is also <u>compact</u>.

In the following we assume the <u>Riemannian case</u>, so g is <u>positive def.</u> This means that $\langle\ ,\ \rangle$ is also positive def., so that

$$\langle\alpha,\alpha\rangle \geq 0, \quad \text{and } \langle\alpha,\alpha\rangle = 0 \text{ iff } \alpha = 0. \quad \text{(any form } \alpha\text{)}.$$

First note that Δ is Hermitian,

$$\begin{aligned}\langle\alpha, \Delta\beta\rangle &= \langle\alpha, d^+d\beta\rangle + \langle\alpha, dd^+\beta\rangle \\ &= \langle d\alpha, d\beta\rangle + \langle d^+\alpha, d^+\beta\rangle \\ &= \langle d^+d\alpha, \beta\rangle + \langle dd^+\alpha, \beta\rangle \\ &= \langle \Delta\alpha, \beta\rangle.\end{aligned}$$

(More simply, just use the rules of $+$ on products of operators, as in QM).

Next note that Δ is a ~~positive definite~~ <u>nonnegative definite operator</u>, i.e., $\langle\alpha, \Delta\alpha\rangle \geq 0 \ \forall\, \alpha$. Proof is easy:

$$\langle \alpha, \Delta\alpha \rangle = \langle \alpha, d^+ d\alpha \rangle + \langle \alpha, dd^+\alpha \rangle$$
$$= \langle d\alpha, d\alpha \rangle + \langle d^+\alpha, d^+\alpha \rangle \geq 0$$

since both terms are ~~pos~~ ≥ 0. In fact, because $\langle\, , \rangle$ is pos. def., we have more:

$$\langle \alpha, \Delta\alpha \rangle = 0 \quad \text{iff} \quad d\alpha = 0 \text{ and } d^+\alpha = 0.$$

In fact, there is more than this. If $d\alpha = 0$ and $d^+\alpha = 0$, then $d^+d\alpha = 0$ and $dd^+\alpha = 0$, so $\Delta\alpha = 0$. But $\Delta\alpha = 0 \Rightarrow \langle \alpha, \Delta\alpha \rangle = 0 \Rightarrow$ $d\alpha = 0$ and $d^+\alpha = 0$. So altogether we have

$$\boxed{\langle \alpha, \Delta\alpha \rangle = 0 \quad \Leftrightarrow \quad \begin{pmatrix} d\alpha = 0 \text{ and} \\ d^+\alpha = 0 \end{pmatrix} \Leftrightarrow \quad \Delta\alpha = 0}$$

Now we make some definitions.

A form $\omega \in \Omega^r(M)$ is

<u>closed</u>	if	$d\omega = 0$
<u>coclosed</u>	if	$d^+\omega = 0$
<u>exact</u>	if	$\omega = d\psi$, some $\psi \in \Omega^{r-1}(M)$
<u>coexact</u>	if	$\omega = d^+\psi$, some $\psi \in \Omega^{r+1}(M)$
<u>harmonic</u>	if	$\Delta\omega = 0$

Note that by thm. above ω is harmonic if and only if it is both closed and coclosed. Actually there there is a interesting set of relationships among the spaces of the different kinds of forms. Define:

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$C = \{\text{closed } r\text{-forms}\} = Z^r(M)$

$CC = \{\text{coclosed } r\text{-forms}\}$

$E = \{\text{exact } r\text{-forms}\} = B^r(M) \subseteq Z^r(M)$

$CE = \{\text{coexact } r\text{-forms}\}$

$H = \{\text{harmonic } r\text{-forms}\} = \text{Harm}^r(M)$.

Then it turns out that $\Omega^r(M)$ can be decomposed into 3 <u>orthogonal</u> subspaces:

$$\boxed{\Omega^r(M) = E \oplus CE \oplus H}$$

<u>Proof</u>: First show that spaces are orthogonal.

(a) prove $\langle E, CE\rangle = 0$. Let $\alpha = d\psi$, $\beta = d^+\phi$ $(\alpha, \beta \in \Omega^r(M))$.

Then $\langle \alpha, \beta\rangle = \langle d\psi, d^+\phi\rangle = \langle \psi, d^+d^+\phi\rangle = 0$.

(b) prove $\langle E, H\rangle = 0$. Let $\alpha = d\psi$, $\Delta\beta = 0$. Then

$$\langle \alpha, \beta\rangle = \langle d\psi, \beta\rangle = \langle \psi, d^+\beta\rangle = 0$$ since $\Delta\beta = 0 \Rightarrow d^+\beta = 0$.

(c) prove $\langle CE, H\rangle = 0$. Let $\alpha = d^+\psi$, $\Delta\beta = 0$. Then

$$\langle \alpha, \beta\rangle = \langle d^+\psi, \beta\rangle = \langle \psi, d\beta\rangle = 0$$ since $\Delta\beta = 0 \Rightarrow d\beta = 0$.

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Next show that $E \oplus CE \oplus H$ is the entire space $\Omega^r(M)$, by showing that if $\omega \in \Omega^r(M)$ is orthogonal to E, CE, and H, then $\omega = 0$. This is a completeness proof. ~~Let $\alpha \in E$, $\beta \in CE$, $\gamma \in H$~~ Suppose

(a) $\langle \omega, \alpha \rangle = 0 \quad \forall\, \alpha \in E$, i.e., $\forall\, \alpha$ such that $\alpha = d\psi$

(b) and $\langle \omega, \beta \rangle = 0 \quad \forall\, \beta \in CE$, i.e., $\forall\, \beta$ such that $\beta = d^{+}\phi$

(c) and $\langle \omega, \gamma \rangle = 0 \quad \forall\, \gamma \in H$, i.e., $\forall\, \gamma$ such that $\Delta\gamma = 0$.

(a) $\Rightarrow \quad \langle \omega, d\psi \rangle = 0 \quad \forall\, \psi \in \Omega^{r-1}(M)$

$\Rightarrow \langle d^{+}\omega, \psi \rangle = 0 \Rightarrow d^{+}\omega = 0.$

(b) $\Rightarrow \langle \omega, d^{+}\phi \rangle = 0 \quad \forall\, \phi \in \Omega^{r+1}(M)$

$\Rightarrow \langle d\omega, \phi \rangle = 0 \Rightarrow d\omega = 0$

(c) (a) and (b) $\Rightarrow \Delta\omega = 0$, so (c) $\Rightarrow \langle \omega, \omega \rangle = 0$ (by $\gamma = \omega$)

$\Rightarrow \omega = 0.$ QED.

See drawing next page.

Here are more relations. We know that $E \subseteq C$ $\left(B^r(M) \subseteq Z^r(M) \right)$. It turns out that $C = E \oplus H$. Similarly, $CC = CE \oplus H$

Proof that $C = E \oplus H$. Let $\langle \alpha, CE \rangle = 0$, i.e., $\langle \alpha, d^{+}\beta \rangle = 0$, $\forall$ $\beta \in \Omega^{r+1}(M)$. This implies $\langle d\alpha, \beta \rangle = 0$, $\forall\, \beta$, $\Rightarrow d\alpha = 0$. Conversely, $d\alpha = 0 \Rightarrow \langle d\alpha, \beta \rangle = 0$, $\forall\, \beta \Rightarrow \langle \alpha, d^{+}\beta \rangle = 0$, $\forall\, \beta \Rightarrow \langle \alpha, CE \rangle = 0$. So α is orthogonal to all co-exact forms iff α is closed. This means,

$$\boxed{\begin{aligned} C &= E \oplus H \\ \text{Sim.} \quad CC &= CE \oplus H \end{aligned}}$$

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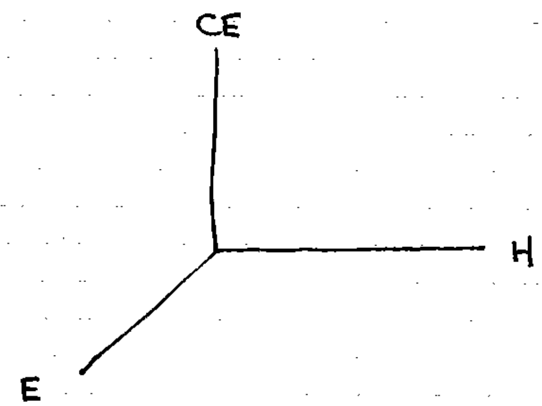
The Hilbert space $\Omega^r(M)$:



C = "E–H plane"

CC = "CE–H plane".

Hence $H = C \cap CC$ as noted above.

From this follows a theorem. An arbitrary form $\omega \in \Omega^r(M)$ has a unique decomposition,

$$\omega = \alpha + \beta + \gamma$$

where $\alpha = d\psi$, $\beta = d^\dagger\phi$, $\Delta\gamma = 0$, i.e.,

$$\omega = d\psi + d^\dagger\phi + \gamma.$$

Finally, there are some connections with cohomology theory. Recall, an element of $H^r(M)$ is an equivalence class $[\omega] = [\omega + d\psi]$ of closed forms, $d\omega = 0$. Look at the geometry of the space $Z^r(M) = C$ before we put in a metric. It's just a vector space, sketch here as 2D, with a subspace of exact forms $E = B^r(M)$, sketch here as a 1D subspace:

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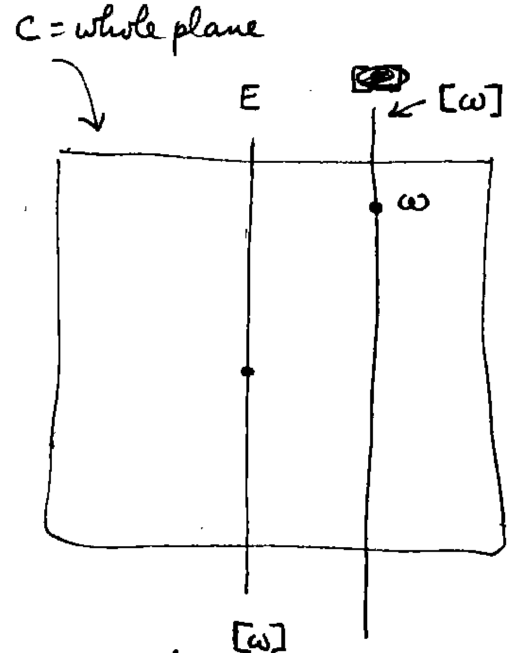
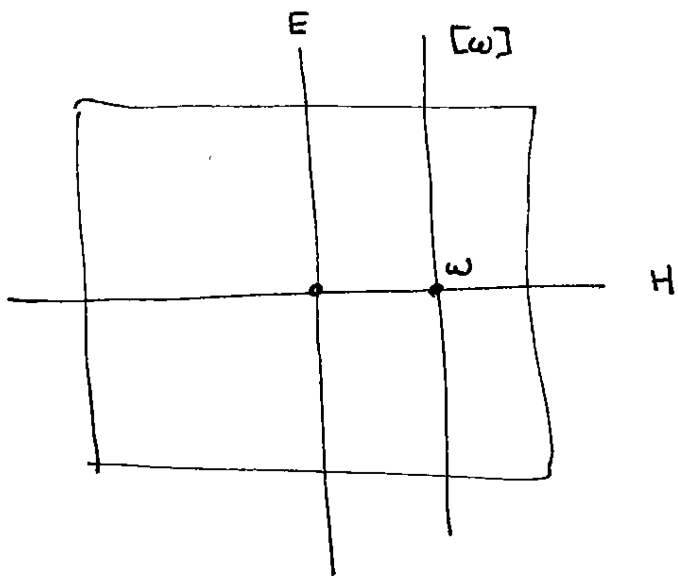


Then an equivalence class is seen geometrically as a plane (line) parallel to E, containing representative element ω, see picture.

Now when we add a metric, we can talk about the orthogonal space in C, which is $H = \mathrm{Harm}^r(M)$:



and there appears a privileged choice for a representative element of ~~an element of $H^r(M)$~~, a cohomology class, namely, a harmonic form:

> Every cohomology class $\in H^r(M)$ contains a unique Harmonic form ω. This is the form in the cohomology class that minimizes $\langle \omega, \omega \rangle$.

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An immediate corollary is that $\mathrm{Harm}^r(M)$ is isomorphic to $H^r(M)$,

$$\boxed{\mathrm{Harm}^r(M) \cong H^r(M)}$$

Every harmonic form corresponds to a unique cohomology class, and vice-versa.

The space of harmonic forms is otherwise the space of eigen-forms of Δ with eigenvalue 0. Thus, the Betti number satisfies

$$b_r = \dim H^r(M) = \text{degeneracy of eigenvalue } 0 \text{ of } \Delta \text{ acting on } \Omega^r(M).$$

Examples. For a zero form, $\Delta f = 0 \Rightarrow df = 0 \Rightarrow f = \text{const}$, and conversely. Thus (assuming M is connected), $\mathrm{Harm}^0(M)$ is spanned by $f=1$, it is one-dimensional, and $b_0(M) = 1$, which we knew already.

For case $r=1$, take some sample spaces.

$S^1 =$ Circle, $\Delta\omega = 0 \Rightarrow \omega = d\theta$. $b_1 = 1$

T^2 Torus, $\Delta\omega = 0 \Rightarrow \omega = d\theta_1, d\theta_2$ $b_1 = 2$

$S^2 =$ sphere $\Delta\omega = 0 \Rightarrow \omega = 0$. $b_1 = 0$.