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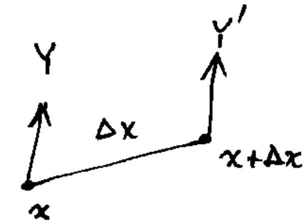
Summary.

Vectors, infinitesimal ∥ transport:



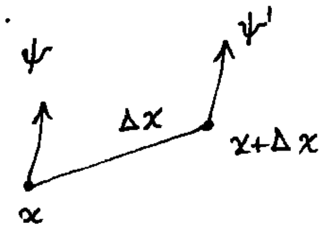
$$Y'^{\mu} = (\delta^{\mu}_{\nu} - \Delta x^{\sigma}\, \Gamma^{\mu}_{\sigma\nu})\, Y^{\nu} \quad \text{(coord)}$$

$$Y'^{\alpha} = \underbrace{(\delta^{\alpha}_{\beta} - \Delta x^{\gamma}\, \Gamma^{\alpha}_{\gamma\beta})}\, Y^{\beta} \quad \text{(O.N. vielbein)}$$

$$= (I + \Omega)^{\alpha}{}_{\beta}, \qquad \Lambda = I + \Omega \in O(3,1)$$

$$\Omega_{\alpha\beta} = -\Delta x^{\gamma}\, \Gamma_{\alpha\gamma\beta} = -\Omega_{\beta\alpha}.$$

Spinors:



$$\psi' = D(\Lambda)\,\psi$$

$D(\Lambda)$ = Dirac "representation" of Lorentz group. Actually, it's not a representation (it's double valued), and it's not a rep. of the whole Lorentz group, only the proper, orthochronous Lorentz group. More on all that in a moment, for now be sloppy and just write $D(\Lambda)$, and call on standard material on Lorentz transforming the Dirac equation. This tells us, for infinitesimal Lorentz transformations, $(\Omega \ll 1)$,

$$D(I + \Omega) = 1 - \frac{i}{4}\, \Omega_{\alpha\beta}\, \sigma^{\alpha\beta},$$

where $\sigma^{\alpha\beta} = \frac{i}{2}[\gamma^{\alpha}, \gamma^{\beta}]$ (standard notation for Dirac matrices). Recall $\{\gamma^{\alpha}, \gamma^{\beta}\} = 2\eta^{\alpha\beta}$

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Summary of the relevant facts regarding Dirac matrices and Lorentz transforming Dirac spinors:

explain later.

$$D(\Lambda_1)\, D(\Lambda_2) = \pm D(\Lambda_1 \Lambda_2)$$

$$\psi' = D(\Lambda)\,\psi \qquad \text{(Lorentz transforming Dirac spinor)}$$

$$D(\Lambda)^{-1}\, \gamma^\alpha\, D(\Lambda) = \Lambda^\alpha{}_\beta\, \gamma^\beta \qquad (\gamma^\alpha \text{ transforms as a 4-vector})$$

$$\gamma^0\, D(\Lambda)^+\, \gamma^0 = D(\Lambda)^{-1}$$

Hence

$$D(I+\Omega) = 1 - \frac{i}{4}\,\Omega_{\alpha\beta}\,\sigma^{\alpha\beta}. \qquad \boxed{\psi' = \left(1 + \frac{i}{4}\,\Delta x^\gamma\, \Gamma_{\alpha\gamma\beta}\, \sigma^{\alpha\beta}\right)\psi}$$

So, basic idea is that under an infinitesimal parallel transport, a spinor transforms by the same (infinitesimal) Lorentz transformation as a vector, but the spinor rep. of the L.T. must be used.

Now, the covariant derivative is defined by the parallel transport. We put $\Delta x^\alpha = \varepsilon X^\alpha$ where $X \in T_x M$, and define (for $Y \in \mathcal{X}(M)$)

$$\nabla_X Y = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\, Y(x+\varepsilon X) - Y' \,\right],$$

gives $(\nabla_X Y)^\mu = X^\nu \left(Y^\mu{}_{,\nu} + \Gamma^\mu_{\nu\sigma}\, Y^\sigma \right).$

Similarly, define (for a spinor <u>field</u> $\psi(x)$):

$$\nabla_X \psi = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\, \psi(x+\varepsilon X) - \psi' \,\right]$$

$$\boxed{\begin{aligned} \nabla_X \psi &= X^\alpha \left[\, \psi_{,\alpha} - \frac{i}{4}\, \Gamma_{\beta\alpha\gamma}\, \sigma^{\beta\gamma}\, \psi \,\right] \\ \nabla_\alpha \psi &= \psi_{,\alpha} - \frac{i}{4}\, \Gamma_{\beta\alpha\gamma}\, \sigma^{\beta\gamma}\, \psi \end{aligned}}$$

Covariant derivs on Dirac spinors

Here $\psi_{,\alpha}$ means $e_\alpha \psi$, e_α = member of O.N. vierbein.

Then Dirac Lagrangian, to be added to gravitational Lagrangian considered above, is

$$\mathcal{L}_D = \bar{\Psi}\left(i\gamma^\alpha \nabla_\alpha - m\right) \psi .$$

We can now show that $\nabla_\alpha \psi$ has the right transformation properties under a gauge transformation. Remind you, latter is defined by

$$e'_\alpha = \Lambda_\alpha{}^\beta e_\beta .$$

Then, given that $\psi' = D(\Lambda)\psi$, you find that

$$\boxed{\nabla'_\alpha \psi' = \Lambda_\alpha{}^\beta \, D(\Lambda) \, \nabla_\beta \psi}$$

Exercise for you to derive

$\nabla_\alpha \psi$ transforms as a covector in its α-index, and as a spinor in ψ. This transformation law is necessary for the invariance of the integral $\int \sqrt{|g|}\, d^4x \, \mathcal{L}_D$.

About the "representation" $D(\Lambda)$. Actually, its a spinor (double-valued) representation, more exactly, should write $\Lambda(D)$ instead of $D(\Lambda)$ if you want a single valued rep. Parallel comparison with the non-relativistic case is useful.

$O(3)$ = classical rotation group, $SO(3)$ = identity component (throw away parity). Then $SU(2)$ = double cover of $SO(3)$, spinor rotation group:

$$\begin{array}{c} SU(2) \\ \Big\downarrow \pi \\ SO(3) \end{array}$$

And then spinors are transformed by some representation of $SU(2)$, $SU(2)$ itself for spin $\frac{1}{2}$ particles, a 4D rep for spin $3/2$, etc.

In the relativistic generalization of this, $O(3,1)$ = full Lorentz group. Denote L_0 = identity component of $O(3,1) = \frac{1}{4}$ of $O(3,1)$ (throw away parity, time reversal). L_0 consists of matrices $\Lambda \in O(3,1)$ such that $\det \Lambda = +1$ and $\Lambda^0{}_0 \geq 1$ (proper, orthochronous Lorentz transformations). The group L_0 has a double cover or spinor representation, which is $SL(2,\mathbb{C})$.

$$\begin{array}{c} L_0 \\ \big\downarrow \pi \\ SL(2,\mathbb{C}) \end{array}$$

So relativistic spinors get transformed by a rep. of $SL(2,\mathbb{C})$. It turns out that there are <u>two</u> inequivalent 2×2 reps of $SL(2,\mathbb{C})$, one is $SL(2,\mathbb{C})$ itself, call the other $\overline{SL(2,\mathbb{C})}$. They correspond to two different ways of promoting the pauli matrices ~~into~~ $\vec{\sigma}$ into a 4-vector,

$$\sigma_\mu = (1, \vec{\sigma}) \quad \text{or} \quad \sigma_\mu = (-1, \vec{\sigma}).$$

The Dirac (4×4) rep. is the direct sum of these two,

$$D = \left[\begin{array}{c|c} SL(2,\mathbb{C}) & 0 \\ \hline 0 & \overline{SL(2,\mathbb{C})} \end{array}\right] \quad \text{in the right basis.}$$

More about this in my 221B notes on the Dirac eqn.

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Now we begin Hodge $*$ theory and harmonic forms. To preview the results a bit, when we add a metric to a manifold we can do new things with differential forms and find new connections to old subjects such as cohomology groups (which do not require a metric for their definition).

If we add a metric to M, we can define a scalar product of wave functions,

$$\cancel{\langle f,g\rangle = \int_M \sqrt{|g|}\, d^m x\, f g, \quad m = \dim M}$$

$$\langle f_1, f_2\rangle = \int_M d^m x \sqrt{|g|}\; f_1 f_2$$

where $m = \dim M$, $f_1, f_2 \in \mathcal{F}(M)$. (Real valued functions here.) Thus the wave functions make a Hilbert space. We also have interesting operators that act on these wave fns, such as the generalized Laplacian ∇^2 (which requires a metric for its definition), and which lead to orthonormal sets of eigenfunctions.

All this (the scalar product, Laplacians) etc. can be generalized to arbitrary r-forms. It turns out for example that the degeneracy of the 0 eigenvalue of ∇^2 is the same as the Betti number of M.

The <u>permutation</u> or <u>Levi-Civita symbol</u> is familiar:

$$\epsilon_{\mu_1 \cdots \mu_m} = \begin{cases} +1 & (\mu_1 \cdots \mu_m) = \text{even pmut of } (1 \dots m) \\ -1 & (\mu_1 \cdots \mu_m) = \text{odd pmut of } (1 \dots m) \\ 0 & \text{otherwise} \end{cases}$$

Just because we put lower indices on it does not mean that it is a tensor. In fact, suppose a tensor has components $\epsilon_{\mu_1 \dots \mu_m}$ in one coord. system x^μ, and examine what its components are in another coord. syst. x'^μ:

Here $\{\theta^\mu\}$ is any basis (coordinate or non-coordinate). Note that if $\{\theta^\mu\}$ is an O.N. vielbein, then $\sqrt{|g|} = 1$ and $\Omega = \theta^1 \wedge \dots \wedge \theta^m$.

It is of interest to compute the completely contravariant components of Ω:

$$\Omega^{\mu_1 \dots \mu_m} = g^{\mu_1 \nu_1} \cdots g^{\mu_m \nu_m} \, \Omega_{\nu_1 \dots \nu_m}$$

$$= \det(g^{\mu\nu}) \, \sqrt{|g|} \, \varepsilon_{\mu_1 \dots \mu_m}$$

But $\det g^{\mu\nu} = \frac{1}{g} = \mathrm{sgn}(g)/|g|$. So,

$$\boxed{\Omega^{\mu_1 \dots \mu_m} = \frac{\mathrm{sgn}(g)}{\sqrt{|g|}} \, \varepsilon_{\mu_1 \dots \mu_m}.}$$ useful later.

We don't worry that LHS has upper indices and RHS has lower, since ε is not a tensor.

Ω is called the <u>invariant volume form</u> since its integral over any region $R \subseteq M$ is the volume of that region in the metrical sense,

$$\int_R \Omega = \mathrm{vol}(R).$$

On a space with $m = \dim M$ dimensions, both r-forms and $(m-r)$-forms have the same number of indep. components,

$$\binom{m}{r} = \binom{m}{m-r}.$$

Thus r-forms and $(m-r)$-forms (at a point $x \in M$) are vector spaces of the same dimensionality, and are isomorphic as vector spaces.

In the absence of a metric or other additional structure, however, there is no natural isomorphism between these spaces. Now, however, we will assume we have a metric (M, g). Then there is a natural mapping $\to$ an isomorphism, actually, between these space,

$$\text{Hodge } * : \Omega^r(M) \to \Omega^{m-r}(M).$$

It is defined by its action on the basis forms of $\Omega^r(M)$, then extended to arb. r-forms by linearity. The defn. is

$$\boxed{*\,(\theta^{\mu_1}\wedge\ldots\wedge\theta^{\mu_r}) = \frac{1}{(m-r)!}\,\Omega^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_{m-r}}\,(\theta^{\nu_1}\wedge\ldots\wedge\theta^{\nu_{m-r}}).}$$

Indices on Ω are raised with $g^{\mu\nu}$.

As a special case, consider the 0-form $1 \in \Omega^0(M)$ (const scalar). Then $r=0$ in the above, and we have

$$*1 = \frac{1}{m!}\,\Omega_{\nu_1\ldots\nu_m}\,\theta^{\nu_1}\wedge\ldots\wedge\theta^{\nu_m} = \Omega,$$

$$\boxed{*1 = \Omega}$$

The defn above makes it clear that $*$ is linear, but is it an isomorphism (i.e., is it invertible)? We answer by computing $**$, a map: $\Omega^r(M) \to \Omega^r(M)$. We apply defn above twice, get

$$**\,(\theta^{\mu_1}\wedge\ldots\wedge\theta^{\mu_r}) = \frac{1}{(m-r)!}\,\Omega^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_{m-r}}$$

$$\times\;\frac{1}{r!}\,\Omega^{\nu_1\ldots\nu_{m-r}}{}_{\lambda_1\ldots\lambda_r}\,(\theta^{\lambda_1}\wedge\ldots\wedge\theta^{\lambda_r}).$$

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Transform this. First raise + lower $\nu_1 \dots \nu_{m-r}$ indices to make indices uniformly upper or lower. Next, on $\Omega_{\nu_1 \dots \nu_{m-r} \lambda_1 \dots \lambda_r}$, migrate λ indices to left of ν indices. This involves $(m-r)r$ sign changes, so

$$\Omega_{\nu_1 \dots \nu_{m-r} \lambda_1 \dots \lambda_r} = (-1)^{r(m-r)} \Omega_{\lambda_1 \dots \lambda_r \nu_1 \dots \nu_{m-r}}.$$

Thus,

$$** (\theta^{\mu_1} \wedge \dots \wedge \theta^{\mu_r}) = \frac{1}{(m-r)!} \frac{1}{r!} (-1)^{r(m-r)}$$

$$\times \boxed{\Omega^{\mu_1 \dots \mu_r \nu_1 \dots \nu_{m-r}} \Omega_{\lambda_1 \dots \lambda_r \nu_1 \dots \nu_{m-r}}} \theta^{\lambda_1} \wedge \dots \wedge \theta^{\lambda_r}$$

$$\rightarrow = \frac{\operatorname{sgn}(g)}{\sqrt{|g|}} \varepsilon_{\mu_1 \dots \mu_r \nu_1 \dots \nu_{m-r}} \times \sqrt{|g|} \, \varepsilon_{\lambda_1 \dots \lambda_r \nu_1 \dots \nu_{m-r}}$$

$$= \operatorname{sgn}(g) \operatorname{sgn}\begin{pmatrix} \mu_1 \dots \mu_r \\ \lambda_1 \dots \lambda_r \end{pmatrix} (m-r)!$$

where we use identities for products of two ε's and where

$$\operatorname{sgn}\begin{pmatrix} \mu_1 \dots \mu_r \\ \lambda_1 \dots \lambda_r \end{pmatrix} = \begin{cases} \pm 1 & \text{if } (\lambda_1 \dots \lambda_r) \text{ is } \binom{\text{even}}{\text{odd}} \text{ perm of } \mu_1 \dots \mu_r \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\operatorname{sgn}\begin{pmatrix} \mu_1 \dots \mu_r \\ \lambda_1 \dots \lambda_r \end{pmatrix} \theta^{\lambda_1} \wedge \dots \wedge \theta^{\lambda_r}$$

$$= r! \; \theta^{\mu_1} \wedge \dots \wedge \theta^{\mu_r}.$$

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Putting it all together, we have

$$**\left(\theta^{\mu_1}\wedge\dots\wedge\theta^{\mu_r}\right) = \mathrm{sgn}(g)\,(-1)^{r(m-r)}\left(\theta^{\mu_1}\wedge\dots\wedge\theta^{\mu_r}\right).$$

or $\boxed{** = \mathrm{sgn}(g)\,(-1)^{r(m-r)}}$ when acting on $\omega\in\Omega^r(M)$.

equivalently, $\boxed{*^{-1} = \mathrm{sgn}(g)\,(-1)^{r(m-r)}\,*}$

Thus $*$ is invertible, and $*$ is an isomorphism.

Now consider the interaction of $*$ with $\wedge$. Let $\alpha,\beta\in\Omega^r(M)$. Then $*\beta$ is an $(m-r)$-form, and

$$\alpha\wedge*\beta \in \Omega^m(M).$$

Thus $\alpha\wedge*\beta$ must be proportional to the volume form Ω, i.e., it must be $f\Omega$ for some scalar f. Now we work out what f is.

Write

$$\alpha = \frac{1}{r!}\alpha_{\mu_1\dots\mu_r}\,\theta^{\mu_1}\wedge\dots\wedge\theta^{\mu_r}$$

$$\beta = \frac{1}{r!}\beta_{\nu_1\dots\nu_r}\,\theta^{\nu_1}\wedge\dots\wedge\theta^{\nu_r}$$

Hence

$$*\beta = \frac{1}{r!}\beta_{\nu_1\dots\nu_r}\frac{1}{(m-r)!}\,\Omega^{\nu_1\dots\nu_r}{}_{\lambda_1\dots\lambda_{m-r}}\,\theta^{\lambda_1}\wedge\dots\wedge\theta^{\lambda_{m-r}},$$

and

$$\alpha\wedge*\beta = \frac{1}{(r!)^2(m-r)!}\alpha_{\mu_1\dots\mu_r}\beta_{\nu_1\dots\nu_r}\,\Omega^{\nu_1\dots\nu_r}{}_{\lambda_1\dots\lambda_{m-r}}$$

$$\times\ \theta^{\mu_1}\wedge\dots\wedge\theta^{\mu_r}\wedge\theta^{\lambda_1}\wedge\dots\wedge\theta^{\lambda_{m-r}}.$$

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Transform this. First raise + lower $\nu_1 \dots \nu_r$ indices, use $\theta^{\mu_1} \wedge \dots \wedge \theta^{\mu_r} \wedge \theta^{\lambda_1} \wedge \dots \wedge \theta^{\lambda_{m-r}} = \varepsilon_{\mu_1 \dots \mu_r \lambda_1 \dots \lambda_{m-r}} \; \theta^1 \wedge \dots \wedge \theta^m$. Get,

$$\alpha \wedge * \beta = \frac{1}{(r!)^2 (m-r)!} \alpha_{\mu_1 \dots \mu_r} \beta^{\nu_1 \dots \nu_r} \sqrt{|g|} \; \varepsilon_{\nu_1 \dots \nu_r \lambda_1 \dots \lambda_{m-r}}$$

$$\times \; \varepsilon_{\mu_1 \dots \mu_r \lambda_1 \dots \lambda_{m-r}} \; \theta^1 \wedge \dots \wedge \theta^m$$

$$= \frac{1}{(r!)^2 (m-r)!} \alpha_{\mu_1 \dots \mu_r} \beta^{\nu_1 \dots \nu_r} (m-r)! \; \mathrm{sgn} \begin{pmatrix} \mu_1 \dots \mu_r \\ \nu_1 \dots \nu_r \end{pmatrix} \Omega$$

$$\boxed{\alpha \wedge * \beta = \left(\frac{1}{r!} \alpha_{\mu_1 \dots \mu_r} \beta^{\mu_1 \dots \mu_r} \right) \Omega}$$

The scalar multiplying Ω is the complete contraction of the components of α with those of β.

Several things to note about this. First, the answer is symmetric in α, β, so

$$\boxed{\alpha \wedge * \beta = \beta \wedge * \alpha}.$$

Next, if g is pos. def., then $\alpha_{\mu_1 \dots \mu_r} \alpha^{\mu_1 \dots \mu_r} \geq 0$, i.e. you get a pos. def. scalar product of r-forms. Define

$$\boxed{\langle \alpha, \beta \rangle = \int_M \alpha \wedge * \beta. \quad = \langle \beta, \alpha \rangle}$$

Then if g is pos. def. (a Riemannian manifold) then this scalar product is also pos. def., i.e., $\langle \alpha, \alpha \rangle \geq 0$ and $\langle \alpha, \alpha \rangle = 0$ iff $\alpha = 0$.

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Notice that if α, β are 0-forms (call them f_1, f_2), then we get the obvious scalar product of them,

$$\langle f_1, f_2 \rangle = \int \Omega\, f_1 f_2 = \int d^m x \sqrt{|g|}\; f_1 f_2 .$$

More generally, if g is pos. def., we have a scalar product on $\Omega^r(M)$ that allows us to define a Hilbert space of r-forms. The functional analysis of this is easiest in the case of compact M.

An example of this scalar product. Let F be the EM field tensor in 4D space time, (maybe curved),

$$F = \frac{1}{2} F_{\mu\nu}\, \theta^\mu \wedge \theta^\nu .$$

then

$$\langle F, F \rangle = \int F \wedge * F = \frac{1}{2} \int F_{\mu\nu} F^{\mu\nu} \sqrt{|g|}\, d^4 x .$$

This is $2\times$ the EM field action,

$$S_{EM} = \frac{1}{2} \langle F, F \rangle .$$

(But the scalar product is not pos. def. on space-time.)

Now consider interaction of $*$ with exterior deriv. d.

Let $\alpha \in \Omega^r(M)$, $\beta \in \Omega^{r-1}(M)$. Then $\langle \alpha, d\beta \rangle$ is meaningful.

We define the operator $d^\dagger$ (the adjoint of d) by

$$\langle \alpha, d\beta \rangle = \langle d^\dagger \alpha, \beta \rangle , \qquad \forall\, \alpha \in \Omega^r(M),\ \beta \in \Omega^{r-1}(M).$$

$d^\dagger$ is the unique operator that makes this equation true. Note that

$$d^\dagger : \Omega^r(M) \to \Omega^{r-1}(M)$$
$$d : \Omega^{r-1}(M) \to \Omega^r(M)$$

(d and $d^\dagger$ work in opposite directions).

We can find an expression for $d^\dagger$ as follows.

$$\langle d^\dagger \alpha, \beta \rangle = \langle \alpha, d\beta \rangle = \langle d\beta, \alpha \rangle = \int_M d\beta \wedge * \alpha.$$

But $d(\beta \wedge * \alpha) = d\beta \wedge * \alpha + (-1)^{r-1} \beta \wedge d * \alpha$, so

$$= \int_M d(\beta \wedge * \alpha) - (-1)^{r-1} \int_M \beta \wedge d * \alpha.$$

First term vanishes by Stokes' thm (we assume $\partial M = 0$), so

$$= (-1)^r \int_M \beta \wedge d * \alpha = (-1)^r \int_M \beta \wedge * (*^{-1} d * \alpha)$$

$$= (-1)^r \langle \beta, *^{-1} d * \alpha \rangle = (-1)^r \langle *^{-1} d * \alpha, \beta \rangle.$$

This implies,

$$\boxed{d^\dagger = (-1)^r *^{-1} d *}$$ acting on r-forms.

In this expression, $*^{-1}$ acts on an $(m-r+1)$-form, so

$$*^{-1} = \mathrm{sgn}(g)\, (-1)^{(m-r+1)(r-1)} *.$$

Since

$$r + (m-r+1)(r-1) \equiv mr + m + 1 \pmod 2,$$

we have

$$\boxed{d^\dagger = \mathrm{sgn}(g)\, (-1)^{mr+m+1} * d *}$$ alternative expression, acting on r-forms.

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We note the identity,

$$\boxed{d^\dagger d^\dagger = 0}$$

which is easily proved,

$$d^\dagger d^\dagger = *d**d* = \underset{\text{sign of } **.}{\pm} *dd* = 0.$$

Note that $d^\dagger$ annihilates any 0-form,

$$d^\dagger f = 0, \qquad f \in \mathcal{F}(M)$$

because there are no (−1)-forms.